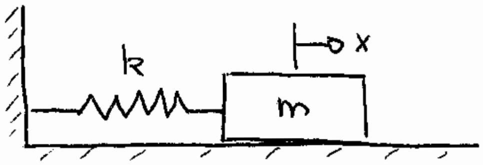
Chap I

Vibration of Single-Degree of Freedom Systems (SDOF)

1. Differential Equation of a Continuous System

Consider a spring-mass oscillator system,



k = unstretched at $t=0$

1.1 Equation of Motion using Newton's 2nd Law

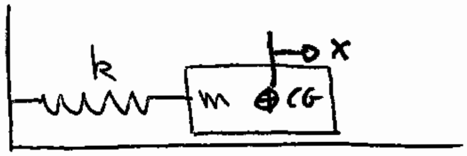
Procedure:

1. Select a suitable coordinate system to describe the motion of the rigid-body.

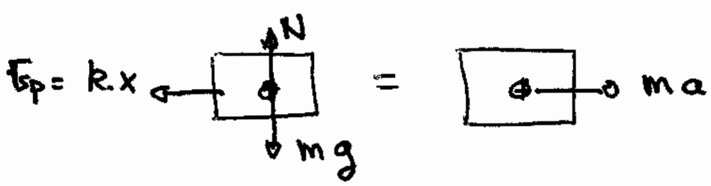
2. Draw the free-body-diagram (FBD)

3. Apply Newton's 2nd Law, $\Sigma \bar{F} = m\bar{a}$, $\Sigma M = I\alpha$

1.



2.



3. $\Sigma \bar{F} = m\bar{a}$ in x-direction

$\Rightarrow \; -k.x = ma$ or $a = \dfrac{d^2x}{dt^2} = \ddot{x}$ (dot indicates differential with respect to time).

$\hookrightarrow \quad m\ddot{x} + kx = 0$

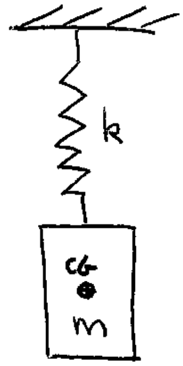
or $\quad \ddot{x} + \dfrac{k}{m} x = 0$

$\Rightarrow \boxed{\ddot{x} + \omega_n^2 x = 0}$ where $\boxed{\omega_n = \sqrt{\dfrac{k}{m}}}$

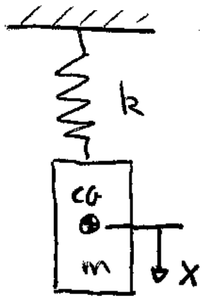
$\omega_n \equiv$ natural frequency (rad/sec)

Exercice 1:

Determine the differential equation of motion

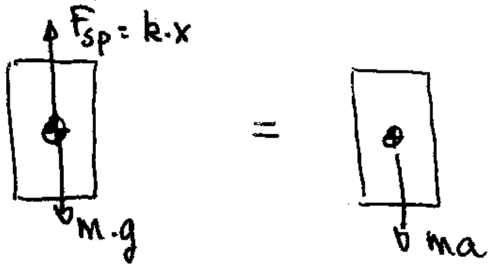


1.



(Not in equilibrium at t=0 sec).

2.



3. Motion $\Rightarrow$ Translation

$\Sigma \bar{F} = m\bar{a}$ in x-direction as shown on figure

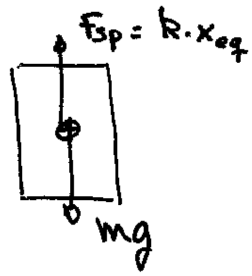
$mg - k.x = ma$ where $a = \frac{d^2x}{dt^2} = \ddot{x}$

$\hookrightarrow \quad m\ddot{x} + kx = m.g$

OR $\quad \ddot{x} + \frac{k}{m}.x = g \Rightarrow \boxed{\ddot{x} + \omega_n^2.x = g}$ - (i)

- Let's rederive the differential equation of motion from the equilibrium position (After spring stretches due to mg).

- At equilibrium ($t=0$) $\Sigma \bar{F} = 0$



$\Sigma \bar{F} = 0 \Rightarrow \quad k.x_{eq} = mg$

$\Rightarrow \boxed{x_{eq} = \frac{m.g}{k}}$

- Let introduce the new variable $\tilde{x}$ that measure the displacement relative to the equilibrium position

$$\boxed{\tilde{x} = x - x_{eq} = x - \frac{m.g}{k}}$$

Substituting $\tilde{x}$ into (i),

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$\ddot{x} + \omega_n^2 . x = g$ and $\tilde{x} = x - \frac{m.g}{k}$ OR $\boxed{x = \tilde{x} + \frac{m.g}{k}}$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega_n^2 \cdot x = g$$

$$\Rightarrow \frac{d^2}{dt^2}\left[\tilde{x} + \frac{m.g}{k}\right] + \omega_n^2\left[\tilde{x} + \frac{m.g}{k}\right] = g$$

$$\Rightarrow \frac{d^2\tilde{x}}{dt^2} + \omega_n^2 \tilde{x} + \omega_n^2 \cdot \frac{mg}{k} = g \quad \text{OR} \quad \omega_n^2 = \frac{k}{m}$$

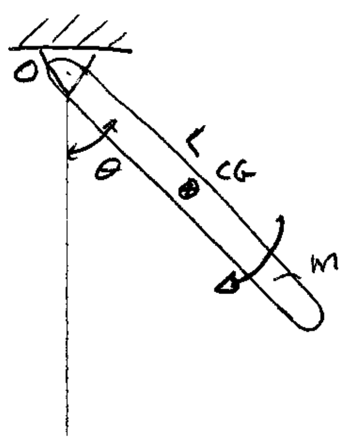
$$\Rightarrow \frac{d^2\tilde{x}}{dt^2} + \omega_n^2 \tilde{x} + \cancel{\frac{k}{m}} \cdot \cancel{\frac{mg}{k}} = \cancel{g}$$

$$\Rightarrow \boxed{\frac{d^2\tilde{x}}{dt^2} + \omega_n^2 \tilde{x} = 0}$$

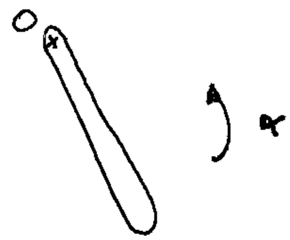
Note: The principal reason for the change of variable $\tilde{x}$ is to obtain the differential equation of motion with no-input load (right-hand side zero). Solution of the type of differential equation is easier than if there is is an input force/load.

Exercice 2 :

Determine the differential equation of motion.



1.



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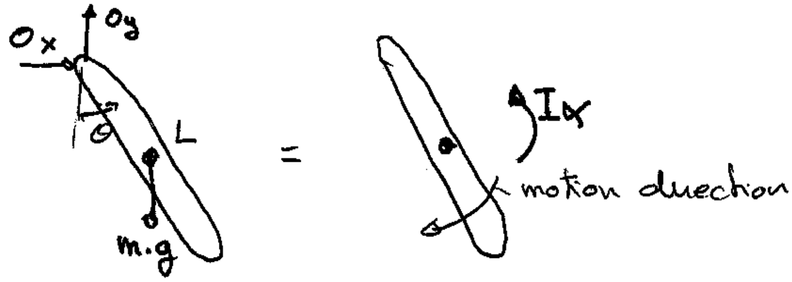


3. Motion type = Rotation $\Rightarrow \Sigma M_o = I_o \cdot \alpha$

$$\circlearrowleft + \; \Sigma M_o = I_o \cdot \alpha$$

$$-mg\frac{L}{2}\cdot\sin\theta = -I_o\cdot\alpha$$

$$\Rightarrow \quad I_o\cdot\alpha - mg\frac{L}{2}\sin\theta = 0$$

$$\begin{cases} \text{where} \quad \alpha = \dfrac{d^2\theta}{dt^2} = \ddot{\theta} \\ \text{and} \quad I_o = \dfrac{mL^2}{3} \end{cases} \text{(Mass Moment of Inertia)}$$

$$\hookrightarrow \quad \frac{mL^{\cancel{2}}}{3}\ddot{\theta} - \cancel{m}g\frac{\cancel{L}}{2}\cdot\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} - \frac{3}{2}\frac{g}{L}\cdot\sin\theta = 0$$

OR $\boxed{\ddot{\theta} - \omega_\theta^2\cdot\sin\theta = 0}$

where $\boxed{\omega_\theta^2 = \frac{3}{2}\frac{g}{L}}$ is the Rotational natural frequency

For small angles $\sin\theta \simeq \theta$,

$$\hookrightarrow \boxed{\ddot{\theta} - \omega_\theta^2 \theta = 0}$$

which is of the same form as the previous equation of motion $\ddot{x} + \omega_n^2 x = 0$.

1.2 Lagrange's Equations to Derive Equations of Motion.

The proof of the Lagrange's method to obtain the equations of motion is derived for the one-dimensional case for the sake of simplicity.

- $\sum \bar{F} = m\bar{a}$ Newton's 2nd law
- $\bar{F} = -\bar{\nabla}.V$ Conservative forces where $V \equiv$ Potential

For 1-D:

Combining the above two equations,

$$\sum \bar{F} = m\bar{a} \text{ and } \bar{F} = -\bar{\nabla}.V$$

$$\Rightarrow \quad -\bar{\nabla}.V = m.\bar{a}$$

$$\Rightarrow \quad -\frac{dV}{dx} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad -\frac{dV}{dx} = m\frac{d}{dt}\left[\frac{1}{2}\frac{d}{d\dot{x}}(\dot{x}^2)\right]$$

Check: $\frac{d}{dt}\left[\frac{1}{2}\frac{d}{d\dot{x}}(\dot{x}^2)\right] = \frac{d}{dt}\left[\frac{1}{\cancel{2}}\cdot\cancel{2}\,\dot{x}\,\frac{d\cancel{\dot{x}}}{\cancel{d\dot{x}}}\right] = \ddot{x} = \frac{d^2x}{dt^2}$ O.K. ✓

Therefore,

$$\hookrightarrow \quad -\frac{dV}{dx} = \frac{d}{dt}\left[\frac{1}{2}\frac{d}{d\dot{x}}(m\dot{x}^2)\right] \qquad \dot{x} = \frac{dx}{dt} = v$$

$$\Rightarrow \quad -\frac{dV}{dx} = \frac{d}{dt}\left[\frac{d}{d\dot{x}}\underbrace{\left(\frac{1}{2}mv^2\right)}_{\text{Kinetic Energy, } K}\right]$$

$$\Rightarrow \quad -\frac{dV}{dx} = \frac{d}{dt}\left[\frac{d}{d\dot{x}}(K)\right]$$

Adding one term on each side whose derivative is zero.

$$\hookrightarrow \quad \frac{d}{dx}(K-V) = \frac{d}{dt}\left[\frac{d}{d\dot{x}}(K-V)\right]$$

OR defining $L = K - V$, $\begin{cases} K \equiv \text{kinetic energy} \\ V \equiv \text{potential energy} \end{cases}$

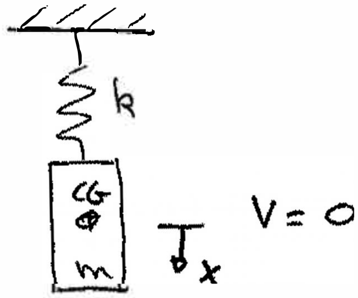
$$\hookrightarrow \quad \frac{dL}{dx} = \frac{d}{dt}\left[\frac{dL}{d\dot{x}}\right]$$

OR $$\boxed{\frac{d}{dt}\left[\frac{dL}{d\dot{x}}\right] - \frac{dL}{dx} = 0}$$ Classical Lagrange's Equation

<u>Exercice 3</u>:

For exercices 1. and 2. rederive the differential equations of motion using the Lagrange's equation.

Exercice 1 $\Rightarrow$



kinetic energy, $K = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$

potential energy, $V = -m g x + \frac{1}{2} k x^2$

$\hookrightarrow \quad L = K - V = \frac{1}{2} m \dot{x}^2 + m . g x - \frac{1}{2} k x^2$

Substituting into the Lagrange's equation

$$\frac{d}{dt}\left[\frac{dL}{d\dot{x}}\right] - \frac{dL}{dx} = 0 \quad - (a)$$

where

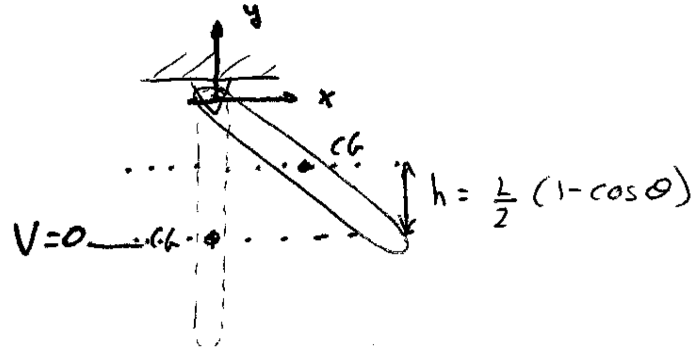
$$\frac{d}{dt}\left[\frac{d}{d\dot{x}}\left(\frac{1}{2} m \dot{x}^2 + m g x - \frac{1}{2} k x^2\right)\right] = m \ddot{x}$$

$$-\frac{dL}{dx} = -\frac{d}{dx}\left(\frac{1}{2} m \dot{x}^2 + m g x - \frac{1}{2} k x^2\right) = -mg + kx$$

$\hookrightarrow$ (a) $\Rightarrow \quad m\ddot{x} + kx = mg$

or $\quad \ddot{x} + \frac{kx}{m} = g \quad$ or $\quad \boxed{\ddot{x} + \omega_n^2 x = g}$

Exercice 2 $\Rightarrow$



- kinetic energy $= \frac{1}{2}\cancel{m}\cancel{v}^{2} + \frac{1}{2} I_o \omega^2$ if moments taken about 'o'

$= \frac{1}{2} m v_{CG}^2 + \frac{1}{2} I_{CG}\,\omega^2$ if moments taken about 'CG'

- potential energy $= mgh = mg\frac{L}{2}(1-\cos\theta)$

Lagrangian $L = K - V = \frac{1}{2} I_o \dot{\theta}^2 - mg\frac{L}{2}(1-\cos\theta)$

OR $L = \frac{1}{2} I_o \dot{\theta}^2 - mg\frac{L}{2}(1-\cos\theta)$

Substituting into the Lagrange's equation

$$\frac{d}{dt}\left[\frac{dL}{d\dot{\theta}}\right] - \frac{dL}{d\theta} = 0 \quad -(i)$$

where

$$\frac{d}{dt}\left[\frac{d}{d\dot{\theta}}\left(\frac{1}{2} I_o \dot{\theta}^2 - mg\frac{L}{2}(1-\cos\theta)\right)\right] = I_o\ddot{\theta}$$

$$\frac{dL}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2} I_o \dot{\theta}^2 - mg\frac{L}{2}(1-\cos\theta)\right] = -mg\frac{L}{2}\sin\theta$$

$\hookrightarrow$ (i) $\quad I_o\ddot{\theta} - mg\frac{L}{2}\sin\theta = 0$

OR Recalling that,

$$I_o = \frac{mL^2}{3}$$

$$\sin\theta \simeq \theta \text{ for small angles}$$

$$\Rightarrow \boxed{\ddot{\theta} - \frac{3}{2}\frac{g}{L}\theta = 0 \quad \text{OR} \quad \ddot{\theta} - \omega_\theta^2 \cdot \theta = 0}$$

where $\omega_\theta = \sqrt{\frac{3}{2}\frac{g}{L}}$

ω_θ is the Rotational natural frequency

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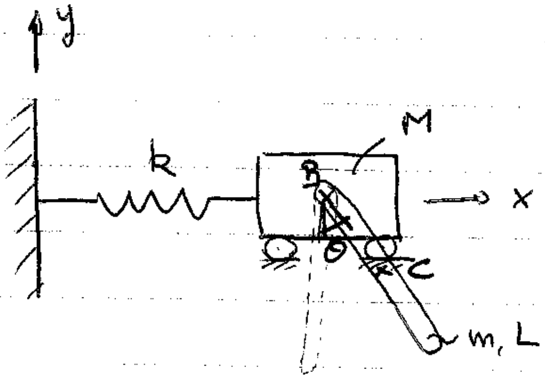
Exercice 4:



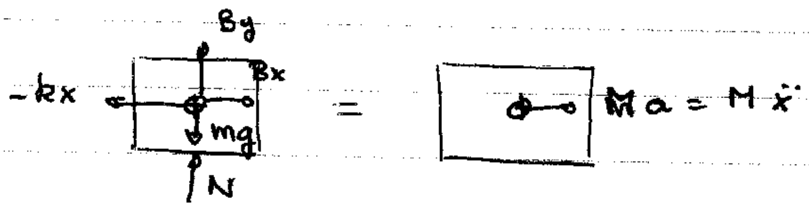
Determine the equations of motion using,

a- Newton's equations

b- Lagrange's equation

Partal

- Block M ⇒ Translation ⇒ $\Sigma \vec{F} = m\vec{a}$

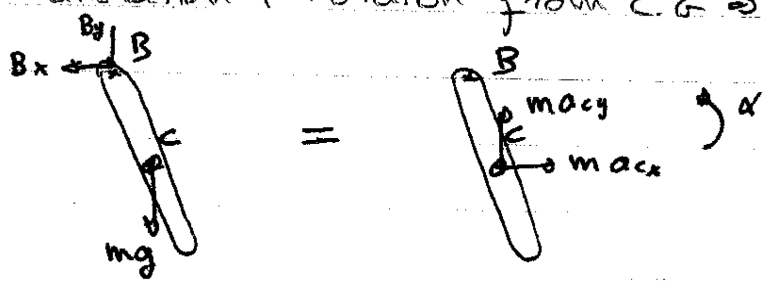


$\Sigma F_x = ma_x \Rightarrow \boxed{-k.x + B_x = M\ddot{x}}$ — ①

$\Sigma F_y = m\cancel{a_y}^{0} \Rightarrow$ not needed

$\circlearrowleft + \Sigma M_B = I\alpha \Rightarrow 0 = 0$ useless

- Bar m ⇒ Rotation from point B + Translation ⇒ Difficult
  OR Translation + Rotation from C.G ⇒ selected



$$\Sigma F_x = m a_{cx} \Rightarrow \boxed{-B_x = m a_{cx}} \quad -(2)$$

$$\Sigma F_y = m a_{cy} \Rightarrow \boxed{-B_y - mg = m a_{cy}} \quad -(3)$$

$$\circlearrowleft + \ \Sigma M_{CG} = I_{CG} \cdot \alpha$$

$$\Rightarrow \boxed{B_x \cdot \frac{L}{2}\cos\theta + B_y \frac{L}{2}\sin\theta = I_{CG} \cdot \alpha} \quad -(4)$$

where $I_{CG} = \dfrac{mL^2}{12}$

- <u>Kinematics</u>

$$\bar{a}_C = \bar{a}_B + \bar{\alpha} \times \bar{R}_{C/B} - \omega^2 \bar{R}_{C/B}$$

$$\Rightarrow \bar{a}_C = \begin{pmatrix} \ddot{x} \\ 0 \\ 0 \end{pmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \ddot{\theta} \\ \frac{L}{2}\sin\theta & -\frac{L}{2}\cos\theta & 0 \end{vmatrix} - \dot{\theta}^2 \begin{pmatrix} L/2 \sin\theta \\ -L/2 \cos\theta \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{a_{cx} = \ddot{x} + L/2 \cos\theta \cdot \ddot{\theta} - \dot{\theta}^2 L/2 \sin\theta} \quad -(5)$$

$$\boxed{a_{cy} = \frac{L}{2}\sin\theta \cdot \ddot{\theta} + \dot{\theta}^2 L/2 \cos\theta} \quad -(6)$$

Substituting (5) and (6) into (2) and (3), respectively

$$-B_x = m a_{cx} \Rightarrow -B_x = m(\ddot{x} + L/2 \cos\theta \cdot \ddot{\theta} - \dot{\theta} L/2 \sin\theta)$$

$$\text{OR} \quad \boxed{B_x = m(-\ddot{x} - L/2 \cos\theta \cdot \ddot{\theta} + \dot{\theta}^2 L/2 \sin\theta)} \quad -(i)$$

$$-B_y - mg = m a_{cy}$$

$$\Rightarrow B_y = -m(g + a_{cy})$$

$$\Rightarrow \boxed{B_y = -m\left(g + \frac{L}{2}\sin\theta.\ddot{\theta} + \dot{\theta}^2 L/2 \cos\theta\right)} - (ii)$$

- Substituting (i) into ①

$$-kx + B_x = M\ddot{x}$$

$$\Rightarrow -kx + m(-\ddot{x} - L/2.\cos\theta.\ddot{\theta} + L/2.\sin\theta.\dot{\theta}^2) = M\ddot{x}$$

$$\Rightarrow \boxed{(m+M)\ddot{x} + k.x + \frac{mL}{2}\left(\ddot{\theta}.\cos\theta - \dot{\theta}^2.\sin\theta\right) = 0}$$

Equation of Motion for the Block.

- Next, substituting (i) and (ii) into ④,

$$B_x \frac{L}{2}\cos\theta + B_y \frac{L}{2}\sin\theta = I_{CG}.\cancel{\alpha}\,\ddot{\theta}$$

$$\Rightarrow m\frac{L}{2}\cos\theta\left(-\ddot{x} - \frac{L}{2}\cos\theta.\ddot{\theta} + \frac{L}{2}\cancel{\sin\theta.\dot{\theta}^2}\right) \dots$$

$$+ \left(-m\frac{L}{2}\right)\sin\theta\left(g + \frac{L}{2}\sin\theta.\ddot{\theta} + \frac{L}{2}\cancel{\cos\theta.\dot{\theta}^2}\right) = \frac{mL^2}{12}\ddot{\theta}$$

$$\Rightarrow -m\frac{L}{2}\cos\theta.\ddot{x} - \frac{mL^2}{4}\left(\cancel{\cos^2\theta + \sin^2\theta}^{\,1}\right)\ddot{\theta} - \frac{mL}{2}g.\sin\theta = \frac{mL^2}{12}\ddot{\theta}$$

Dividing by $\frac{mL}{2}\cos\theta$

$$\hookrightarrow -\ddot{x} - \frac{\frac{\cancel{m}L^2}{4}}{\frac{\cancel{mL}}{2}\cos\theta}.\ddot{\theta} - g.\tan\theta = \frac{\frac{\cancel{m}L^2}{12}}{\frac{\cancel{mL}}{2}\cos\theta}\ddot{\theta}$$

$$\Rightarrow \quad \ddot{x} + \ddot{\theta}\left(\frac{L}{6\cos\theta} + \frac{L}{2\cos\theta}\right) + g\tan\theta = 0$$

$$\Rightarrow \quad \boxed{\ddot{x} + \frac{2L}{3\cos\theta}\cdot\ddot{\theta} + g\tan\theta = 0}$$

Equation of Motion for the Bar

Part b)

- Potential energy, $V = \underbrace{\frac{1}{2}kx^2}_{\text{spring}} + \underbrace{mg\frac{L}{2}(1-\cos\theta)}_{\text{Bar}}$

- kinetic energy, $K = \frac{1}{2}MV_B^2 + \frac{1}{2}mV_C^2 + \frac{1}{2}I_{CG}\omega^2$ $(CG = C)$

- Lagrangian $L = K - V$

$$\Rightarrow L = \frac{1}{2}MV_B^2 + \frac{1}{2}mV_C^2 + \frac{1}{2}I_{CG}\omega^2 - \frac{1}{2}kx^2 - \frac{mgL}{2}(1-\cos\theta)$$

OR

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}mV_C^2 + \frac{1}{2}I_{CG}\dot{\theta}^2 - \frac{1}{2}kx^2 - \frac{mgL}{2}(1-\cos\theta)$$

where • $I_{CG} = \frac{mL^2}{12}$

• $\bar{V}_C = \bar{V}_B + \bar{\omega} \times \bar{R}_{C/B}$

$$\Rightarrow \bar{V}_C = \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ \frac{L}{2}\sin\theta & -\frac{L}{2}\cos\theta & 0 \end{vmatrix}$$

$$\Rightarrow \quad \bar{V}_c = \begin{pmatrix} \dot{x} + \frac{L}{2}\cos\theta \cdot \dot{\theta} \\ \frac{L}{2}\sin\theta \cdot \dot{\theta} \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{V_c = |\bar{V}_c| = \sqrt{\left(\dot{x} + \frac{L}{2}\cos\theta \cdot \dot{\theta}\right)^2 + \left(\frac{L}{2}\sin\theta \cdot \dot{\theta}\right)^2}}$$

- The Lagrangian becomes,

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[\left(\dot{x} + \frac{L}{2}\cos\theta \cdot \dot{\theta}\right)^2 + \left(\frac{L}{2}\sin\theta \cdot \dot{\theta}\right)^2 \right] + \frac{1}{2} I_c \dot{\theta}^2 \ldots$$

$$- \frac{1}{2} k x^2 - mg \frac{L}{2}(1 - \cos\theta)$$

$$\Rightarrow L = \frac{1}{2} M \dot{x}^2 + \frac{m}{2}\left[\dot{x}^2 + \frac{L^2}{4}\cos^2\theta \cdot \dot{\theta}^2 + L\dot{x}\cos\theta \cdot \dot{\theta} + \frac{L^2}{4}\sin^2\theta \cdot \dot{\theta}^2\right] + \frac{1}{2} I_c \cdot \dot{\theta}^2 \ldots$$

$$- \frac{1}{2} k x^2 - mg \frac{L}{2}(1 - \cos\theta)$$

$$\Rightarrow L = \frac{1}{2}\left[(m+M)\dot{x}^2 + \frac{mL^2}{4}\dot{\theta}^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + mL\dot{x}\cos\theta \cdot \dot{\theta}\right] + \frac{1}{2}\left(\frac{mL^2}{12}\right)\dot{\theta}^2 \ldots$$

$$- \frac{1}{2} k x^2 - m g \frac{L}{2}(1 - \cos\theta)$$

$$\Rightarrow \boxed{L = \frac{1}{2}\left[(m+M)\dot{x}^2 + mL\dot{x} \cdot \dot{\theta}\cos\theta + \frac{mL^2}{3}\dot{\theta}^2\right] - \frac{1}{2} k x^2 - mg\frac{L}{2}(1 - \cos\theta)}$$

- Substituting into the Lagrange's equation

$$\boxed{\frac{d}{dt}\left[\frac{dL}{d\dot{x}}\right] - \frac{dL}{dx} = 0}$$

$$\frac{d}{dt} = \frac{d}{d\theta} \cdot \frac{d\theta}{dt}$$

- $$\frac{\partial L}{\partial \dot{x}} = (m+M)\dot{x} + \frac{m}{2}L\dot{\theta}\cdot\cos\theta$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = (m+M)\ddot{x} + \frac{mL}{2}\left(\ddot{\theta}\cos\theta - \sin\theta\cdot\dot{\theta}^2\right)$$

- $$\frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow (m+M)\ddot{x} + \frac{mL}{2}\left(\ddot{\theta}\cos\theta - \sin\theta\cdot\dot{\theta}^2\right) + kx = 0$$

OR

$$\boxed{(m+M)\ddot{x} + kx + \frac{mL}{2}\left(\ddot{\theta}\cos\theta - \sin\theta\cdot\dot{\theta}^2\right) = 0}$$

as previously see p: 14

Equation of Motion for the Block

Similarly,

- $$\frac{\partial L}{\partial \dot{\theta}} = \frac{mL}{2}\dot{x}\cos\theta + \frac{1}{3}mL^2\dot{\theta}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = \frac{mL}{2}\ddot{x}\cos\theta - \frac{mL}{2}\dot{x}\dot{\theta}\sin\theta + \frac{mL^2}{3}\ddot{\theta}$$

- $$\frac{\partial L}{\partial \theta} = -\frac{mL}{2}\dot{x}\dot{\theta}\sin\theta - mg\frac{L}{2}\sin\theta$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{mL}{2}\ddot{x}\cos\theta + \frac{mL^2}{3}\ddot{\theta} + mL\dot{x}\underbrace{\cancel{\dot{\theta}(\sin\theta - \sin\theta)}}_{0} + mg\frac{L}{2}\sin\theta = 0$$

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Dividing by $\frac{mL}{2}\cos\theta$,

$$\hookrightarrow \quad -\ddot{x} + \frac{\frac{\cancel{m}L^{\cancel{2}}}{3}}{\frac{\cancel{m}\cancel{L}}{2}\cos\theta}\ddot{\theta} + \frac{\frac{\cancel{m}g\cancel{L}}{2}\sin\theta}{\frac{\cancel{m}\cancel{L}}{2}\cos\theta} = 0$$

$$\Rightarrow \quad \boxed{\ddot{x} - \frac{2L}{3\cos\theta}\ddot{\theta} + g\tan\theta}$$ as previously (see p:15)

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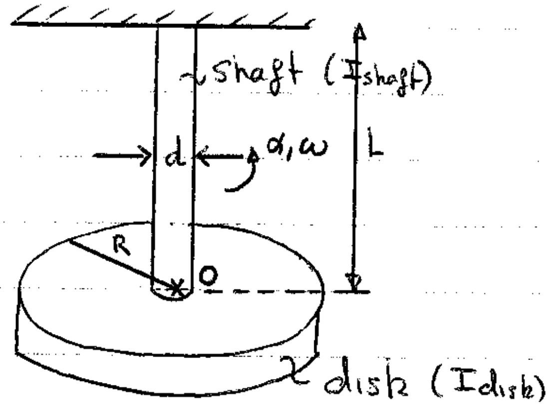
2. Free Vibration of a Torsional system

Consider the following system,

$I_{shaft})_{CG} = \frac{mL^2}{12}$

$I_{disk})_{CG} = \frac{mR^2}{2}$

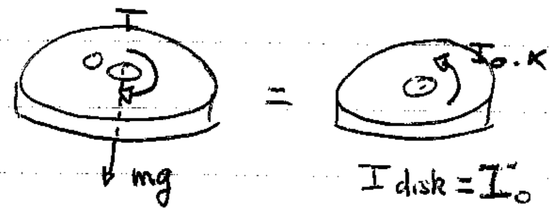


Determine the equation of motion of the disk?

Free-Body Diagram:

Motion ⇒ Rotation

$\Sigma M = I\alpha$.



$\circlearrowleft + \Sigma M_o = I_o \alpha \Rightarrow \boxed{-T = I_o \ddot{\theta}}$ – (i)

- Recalling from solid-mechanics

<u>Shaft</u>: $\frac{d\theta}{dx} = \frac{T}{GJ} \Rightarrow \theta = \frac{TL}{GI_{shaft}}$

Area moment of inertia

$I_{shaft} = \frac{\pi d^4}{32}$

$\hookrightarrow T = \frac{GI_{shaft}.\theta}{L} = \frac{G\pi d^4.\theta}{32L}$

$\Rightarrow \boxed{T = \frac{G\pi d^4}{32L}\theta}$ – (ii)

Combining (i) and (ii),

$$-T = I_{dish}\,\ddot{\theta}$$

$$\Rightarrow \quad T + I_{dish}\,\ddot{\theta} = 0$$

$$\Rightarrow \quad I_{dish}\,\ddot{\theta} + \frac{G\pi d^4}{32L}\,\theta = 0$$

where $I_{dish} = \dfrac{mR^2}{2}$ $\quad$ m = mass of dish.

$$\hookrightarrow \quad \frac{mR^2}{2}\,\ddot{\theta} + \frac{G\pi d^4}{32L}\,\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \frac{G\pi d^4}{32L}\cdot\frac{2}{mR^2}\cdot\theta = 0$$

$$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{G\pi d^4}{16LmR^2}\,\theta = 0}$$

OR $\boxed{\ddot{\theta} + \omega_\theta^2\cdot\theta = 0}$

where $\boxed{\omega_\theta = \sqrt{\dfrac{G\pi d^4}{16LmR^2}}}$

<u>check units:</u> $\dfrac{G\pi d^4}{16LmR^2} = \dfrac{\frac{N}{m^2}\cdot m^4}{\cancel{m}\cdot\frac{N\cdot sec^2}{\cancel{m}}\,m^2} = \dfrac{1}{sec^2}$ ✓✓ O.K

Using the Lagrange's equation:

$$\begin{cases} \text{Kinetic energy} & K = \frac{1}{2} I_o \omega^2 \\ \text{Potential energy} & V = \frac{1}{2} k_\theta \theta^2 \end{cases}$$, k_θ = torsional spring constant.

Lagrangian's $\Pi = K - U$

$$\Rightarrow \Pi = \frac{1}{2} I_o \omega^2 - \frac{1}{2} k_\theta \theta^2$$

$$\Rightarrow \Pi = \frac{1}{2} I_o \dot{\theta}^2 - \frac{1}{2} k_\theta \theta^2$$

Substituting into the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \boxed{I_o \ddot{\theta} + k_\theta \theta = 0} - (a)$$

$$I_o = \frac{mR^2}{2}$$

$$T = k_\theta \theta = GJ \frac{d\theta}{dx}$$

$$\Rightarrow k_\theta . \cancel{\theta} = \frac{GJ \cancel{\theta}}{L} \Rightarrow k_\theta = \frac{GJ}{L} \text{ where } J = \frac{\pi}{32} d^4$$

$$\hookrightarrow k_\theta = \frac{G\pi}{32L} d^4$$

The equation (a) becomes

$$\frac{mR^2}{2} \ddot{\theta} + \frac{G\pi}{32L} d^4 \theta = 0$$

$$\Rightarrow \boxed{\ddot{\theta} + \frac{G\pi d^4}{16 L m R^2} \theta = 0}$$ which is similar to the equation in p: 20

3 Damping

The energy generated by the vibrational motion is gradually converted to heat or sound. Due to the reduction in energy the response (displacement) of the system gradually decreases. The mechanism by which the vibrational energy is converted into heat or sound is known as damping.
A damping force exists only if there is relative velocity between the two ends of the damper. Mathematically, the damping is modelled as a force synchronuous with the velocity but opposite in direction.

It is difficult to determine the causes of damping in practical cases. In general damping can be modelled as **Viscous damping**, **Coulomb** or **Dry Friction damping**, and **Hysteretic damping**.

3.1 Viscous Damping

Viscous damping is the most common used damping mechanism. When a mechanical system vibrate in a fluid medium such as air, gas, water, and oil, the resistance offered by the fluid medium to the moving body causes energy to be dissipated. In viscous damping, the damping force is proportional to the velocity of the vibrating body.

Examples: 1)- Fluid flow around a journal in a bearing.
2) Fluid flow around a piston in a cylinder
3) Fluid flow between sliding surfaces

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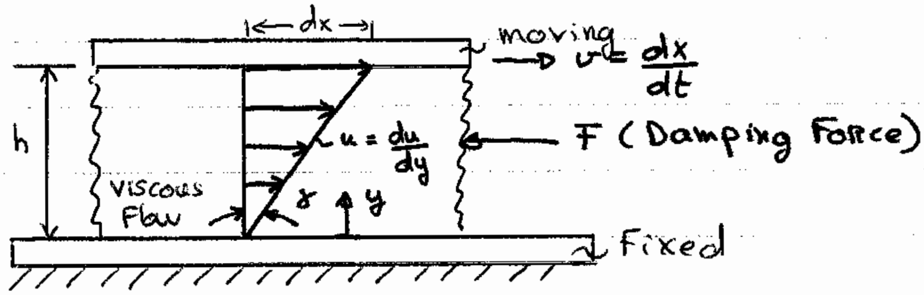
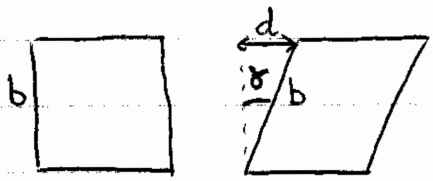
Exercice 5: Construction of Viscous Dampers

A viscous damper can be constructed using two parallel plates separated by a distance h, with a fluid of viscocity μ.

$\rho \equiv$ density
$\mu \equiv$ dynamic viscocity
$\nu \equiv$ kinematic viscocity
$\nu = \frac{\mu}{\rho}$



- Recalling from solid-mechanics the concept of shear:



$\tan\gamma \approx \sin\gamma \simeq \frac{d}{b} \Rightarrow \gamma = \frac{d}{b}$

$\gamma \equiv$ shear strain

↳ Shear stress, $\tau = \underset{\sim}{G}\,\gamma$
Constants (Modulus of Rigidity)

- Recalling from fluids, $\tan(\gamma) = \frac{u}{h} = \frac{du/dy}{h} = \left(\frac{v}{h^2}\right) y$

$$\Rightarrow \gamma \simeq \frac{du}{dy}\frac{1}{h} \simeq \frac{v}{h^2}\cdot y$$

$$\tau = \mu\,\gamma \qquad \mu \equiv \text{Dynamic viscocity}$$

and $\tau = \frac{F}{A} \Rightarrow -\frac{F}{A} = \mu\gamma = \frac{\mu v}{h^2}\,y$

For maximum shear $y = h$

↳ $-\frac{F}{A} = \frac{\mu v}{h^{\cancel{2}}}\cdot\cancel{h}$

$$\Rightarrow \quad -\frac{F}{A} = \frac{\mu v}{h}$$

$$\Rightarrow \quad F = -\underbrace{\left(\frac{\mu A}{h}\right)}_{\text{constant}} v$$

OR $\boxed{F_{damping} = -c \cdot v}$ where $\boxed{c = \frac{\mu A}{h}}$ Damping constant

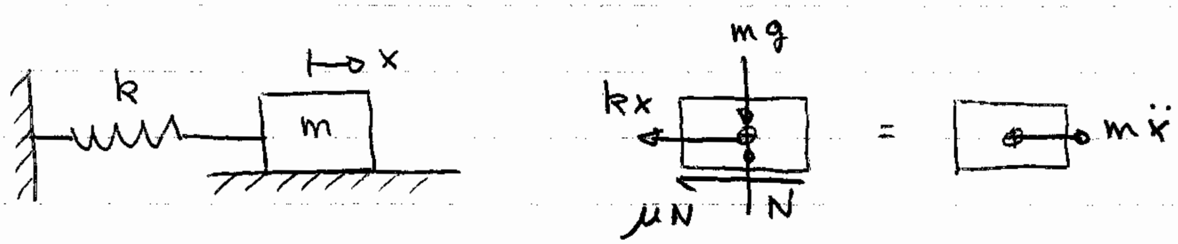
Notes:

- Viscous damping proportional to velocity
- the viscous damping depends on the fluid properties and geometric characteristics.

3.2 Coulomb or Dry-Friction Damping

The damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces

$\boxed{F_{damping} = -\mu N}$ N = Normal force

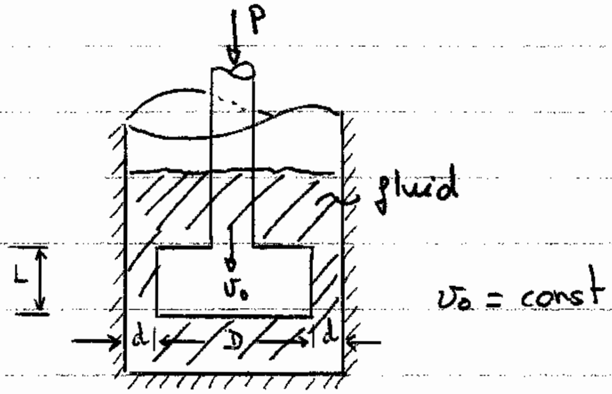


$$\Sigma F_x = m a_x \Rightarrow -kx - \mu N = m\ddot{x}$$

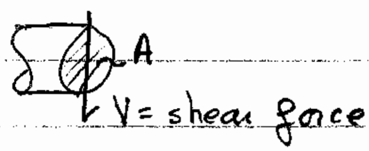
$$\Rightarrow \boxed{m\ddot{x} + kx = \mu N}$$

Exercice 6 (Piston Cylinder Dashpot)

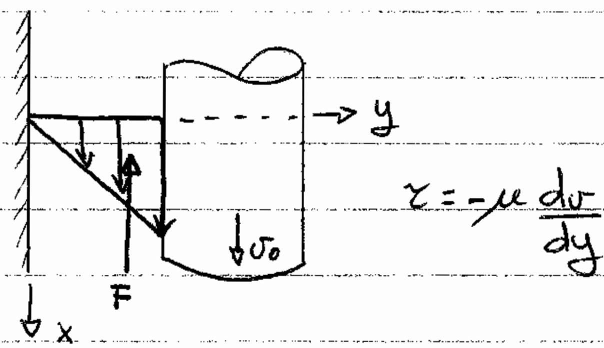
Develop an expression for the damping of the dashpot shown below.



- Shear Force $\tau = \dfrac{\text{Force}}{\text{tangenhal Area}}$



- Recalling from p: 22



- $$F = \pi D L \, d\tau = \pi D L \frac{d\tau}{dy} dy$$

$$\Rightarrow \boxed{F = -\pi D L \mu \frac{d^2 v}{dy^2} dy} \quad -(i)$$

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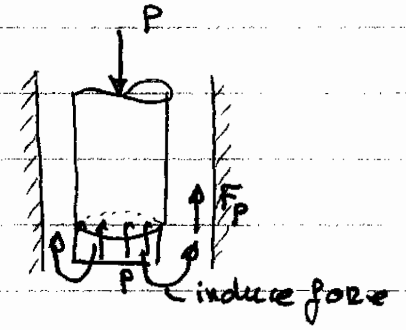
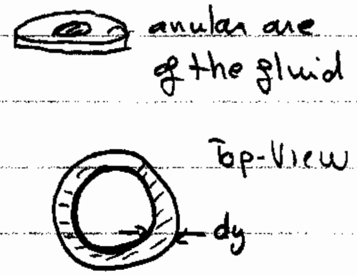
- The force created by the piston will induce a force in the gap between the wall and the piston.

pressure $p = \dfrac{P}{\frac{\pi D^2}{4}}$



$$\Rightarrow \boxed{p = \frac{4P}{\pi D^2}}$$

$$\hookrightarrow \quad F_p = p(\underbrace{\pi D\,dy}_{\text{anular area}}) = \frac{4P\,\cancel{\pi}\,\cancel{D}\,dy}{\cancel{\pi}D^{\cancel{2}}}$$



$$\Rightarrow \boxed{F_p = \frac{4P}{D}\,dy} \quad \text{— (ii)}$$

- Applying Newton's Equations

$\Sigma \bar{F} = m\bar{a}$, $\Sigma F_x = m a_x$ where $a_x = \dfrac{dv_0}{dt} = 0$ since $v_0 = \text{const.}$

$\hookrightarrow \; -F - F_p = 0$

$$\Rightarrow -\pi D L \mu \frac{d^2 v}{dy^2}\,\cancel{dy} = \frac{4P}{D}\,\cancel{dy}$$

$$\Rightarrow \boxed{\frac{d^2 v}{dy^2} = -\frac{4P}{\pi D^2 L \mu}}$$

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Integrating twice with the appropiate boundary conditions.

$$\frac{d^2v}{dy^2} = -\frac{4P}{\pi D L \mu}$$

$$\Rightarrow \int \frac{d^2v}{dy^2} = \int \frac{-4P}{\pi D L \mu} dy$$

$$\Rightarrow \frac{dv}{dy} = \frac{-4P}{\pi D L \mu} y + c_1$$

$$\Rightarrow \boxed{v = \frac{-4P}{\pi D^2 L \mu} \frac{y^2}{2} + c_1 y + c_2} \text{ — (iii)}$$

Boundary Conditions: $\begin{cases} v = v_0 \text{ at } y = d \\ v = 0 \text{ at } y = 0 \end{cases}$

$\hookrightarrow$ $v = 0$ @ $y = 0$, $\boxed{c_2 = 0}$

$v = v_0$ @ $y = d$, $v_0 = \frac{-2P}{\pi D L \mu} d^2 + c_1 d$

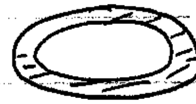
$$\Rightarrow \boxed{c_1 = \frac{v_0}{d} + \frac{2P.d}{\pi D^2 L \mu}}$$

Substituting c_1 and c_2 into (iii),

$$\boxed{v = \frac{2P}{\pi D^2 L \mu}(-y^2 + yd) + \frac{v_0}{d} y}$$

- Next, the flow rate through the gap 'd' can be obtained as follows

TOP-VIEW



$A = \pi D\, dy$

$$\hookrightarrow \quad Q_1 = \int_0^d v\,(\pi D\, dy)$$

$$\Rightarrow \quad Q_1 = \pi \cdot D \left\{ \frac{2P}{\pi D^3 L \mu} \left[-\int_0^d y^2\, dy + d \int_0^d y\, dy \right] + \frac{v_0}{d} \int_0^d y\, dy \right\}$$

$$\Rightarrow \quad Q_1 = \pi \cdot D \left\{ \frac{2P}{\pi D^3 L \mu} \left(\cancelto{d^3/6}{-\frac{d^3}{3} + \frac{d^3}{2}} \right) + \frac{v_0}{d} \frac{d^2}{2} \right\}$$

$$\Rightarrow \quad \boxed{Q_1 = \pi D \left\{ \frac{P d^3}{3 \pi D^3 L \mu} + \frac{v_0 d}{2} \right\}} \; - (a)$$

- The flow rate Q_1 must be equal to the flow rate induced by the piston

$A = \frac{\pi D^2}{4}$

$$\hookrightarrow \quad \boxed{Q_2 = v_0 \cdot A = v_0 \left(\frac{\pi D^2}{4} \right)} \; - (b)$$

- Equating (a) and (b), and solving for P

$$\hookrightarrow \quad v_0 \frac{\cancel{\pi} D^{\cancel{2}}}{4} = \cancel{\pi} \cdot \cancel{D} \left\{ \frac{P d^3}{3 \pi D^3 L \mu} + \frac{v_0 d}{2} \right\}$$

$$\Rightarrow \quad P = \frac{3 \pi D^3 L \mu}{d^3} \left[\frac{D}{4} - \frac{d}{2} \right] v_0$$

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Recalling that the damping force (viscous damping) is of the form

$$\boxed{F_{damping} = c\,v}$$

$$\rightarrow \quad P = \frac{3\pi D^3 L}{2d^3}\mu\left[\frac{D}{2} - d\right] v_0$$

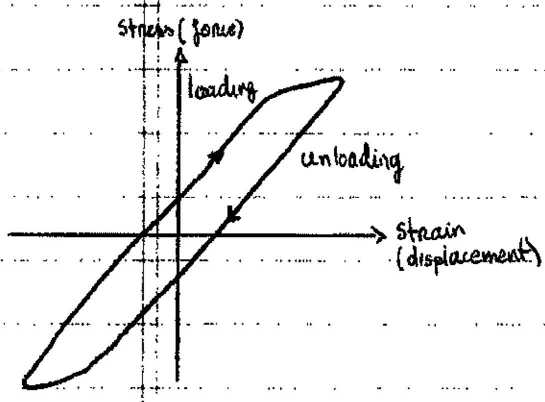
$$\text{or} \quad P = c \cdot v_0$$

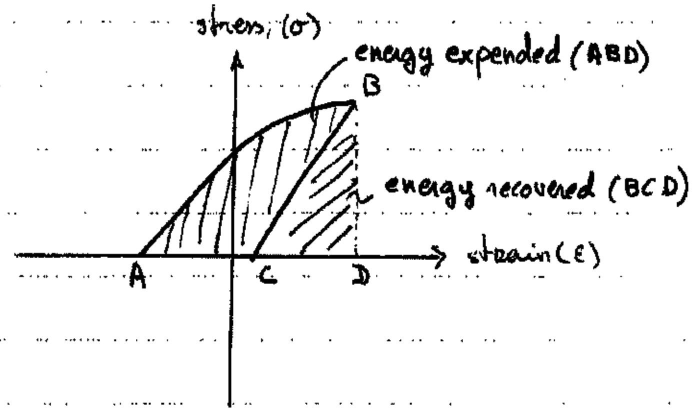
where $c \equiv$ viscous-damping coefficient

$$\boxed{c = \frac{3\pi D^3 L}{2d^3}\mu\left[\frac{D}{2} - d\right]}$$

3.3 Hysteris Loop

When materials are deformed, energy is absorbed and dissipated by the material. The effect is due to friction between the internal planes, which slip or slide as the deformation take place.
When a body having material damping is subjected to vibrations, the stress-strain diagram shows a hysteris loop. The area of the loop denotes the energy lost per unit volume of the body per cycle due to damping.





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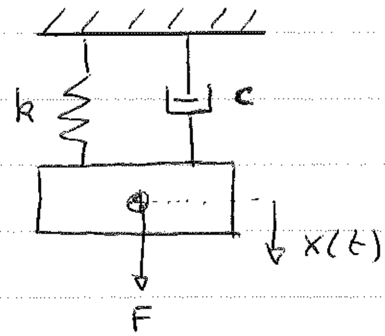
Consider the spring - viscous damper system shown below

The force required to displace the mass is the force required to overcome the spring and damper force.

$$\boxed{F = kx + c\dot{x}}$$



Assuming harmonic solution: $x(t) = X\sin(\omega.t)$

$\Rightarrow \quad F = kx + c\dot{x}$

$\quad\quad = kX\sin(\omega t) + cX\omega\cos(\omega.t)$

Using the trigonometric relation:

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$$

$\Rightarrow$

$$F = k.X\sin(\omega t) + cX\omega.\sqrt{1 - \sin^2(\omega t)}$$

$$\Rightarrow \quad F = kX\sin(\omega.t) + c.\omega\sqrt{X^2 - (X.\sin(\omega t))^2}$$

Since $x = X\sin(\omega.t)$

$\Rightarrow$

$$F = kx + c.\omega\sqrt{X^2 - x^2}$$

$$\Rightarrow \quad F - kx = c.\omega\sqrt{X^2 - x^2}$$

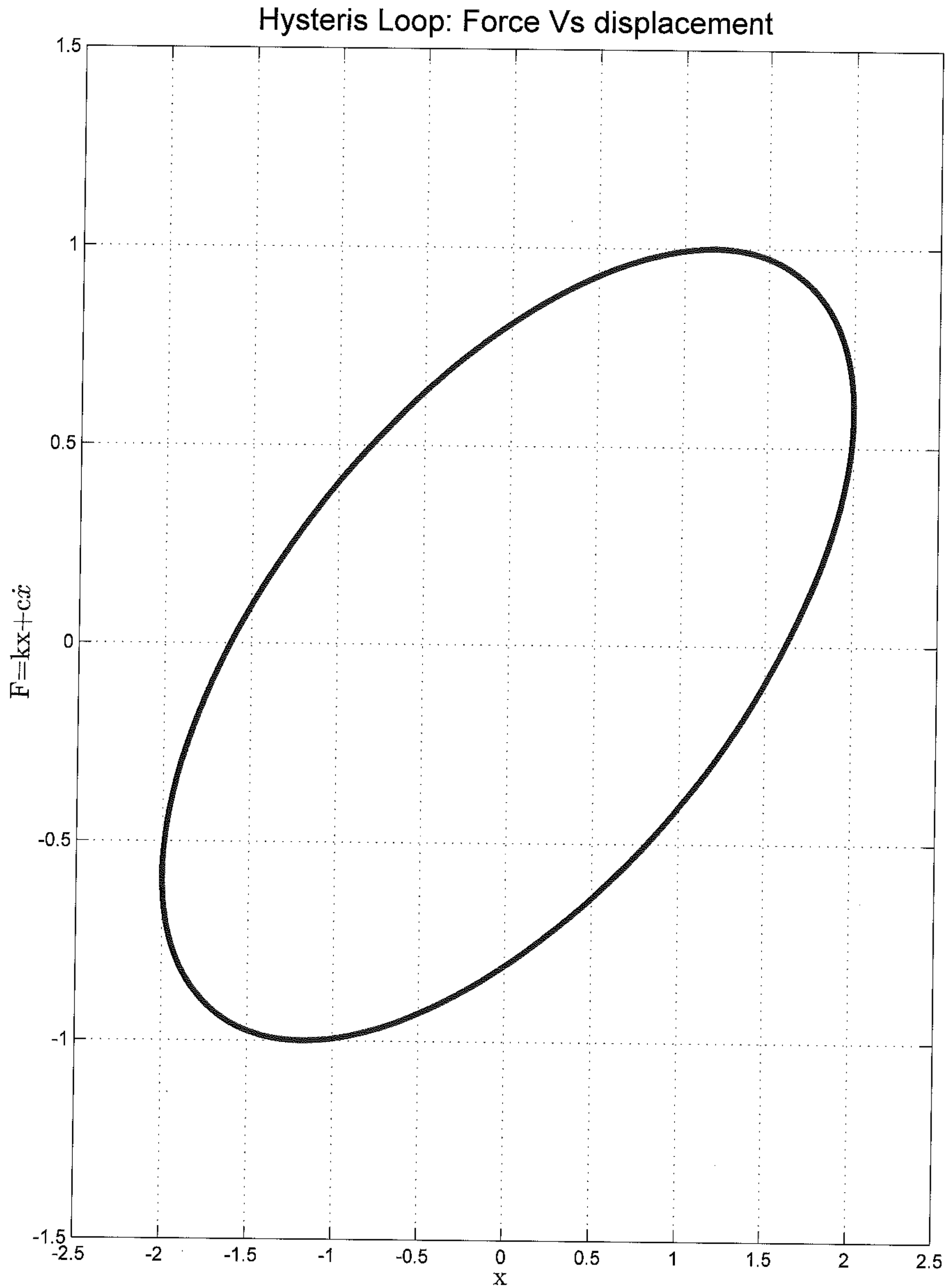
Squaring the above expression,

$$(F - kx)^2 = \left\{c.\omega\sqrt{X^2 - x^2}\right\}^2$$

$$\Rightarrow \quad F^2 - 2Fkx + k^2x^2 = c^2.\omega^2(X^2 - x^2)$$

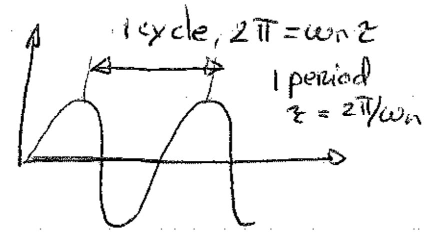
$$\Rightarrow \quad \boxed{F^2 + (k^2 + c^2.\omega^2).x^2 - (2k)xF - c^2.\omega^2 X^2 = 0}$$

which is the general form of the ellipse equation rotated about the origin in the F-x plane



Hysteris Loop: Force Vs displacement
1.5
1
0.5
0
-0.5
-1
-1.5
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
x
F=kx+c$\dot{x}$

The area of the loop represents the energy lost by the damper in a cycle.



$$\Delta E = \oint F.dx$$

$$= \oint (kx + c\dot{x})\, dx$$

Recalling that,

$$x = X \sin(\omega . t)$$

$$\dot{x} = \omega X \cos(\omega . t)$$

and using the chain rule $dx = \frac{dx}{dt} . dt$

$$\Rightarrow \quad \Delta E = \oint F . dx = \oint F \frac{dx}{dt} . dt$$

$$\Rightarrow \quad \Delta E = \oint F . \dot{x}\, dt$$

$$\Rightarrow \quad \Delta E = \int_0^{\frac{2\pi}{\omega}} \left(k . X \sin(\omega . t) + c . \omega X \cos(\omega . t) \right) \omega . X \cos(\omega t) dt$$

$$\Rightarrow \quad \Delta E = \int_0^{2\pi/\omega} k \omega X^2 \sin(\omega . t) . \cos(\omega t)\, dt + \int_0^{2\pi/\omega} c . \omega^2 . X^2 \cos^2(\omega t) dt$$

$$\Rightarrow \quad \Delta E = -\frac{1}{2} k X^2 \cos^2(\omega t) \Big|_0^{\frac{2\pi}{\omega}} + c \omega X^2 \left[\frac{\sin(\omega t) . \cos(\omega t)}{2} + \frac{\omega}{2t} \right|_0^{\frac{2\pi}{\omega}}$$

$$\Rightarrow \quad \Delta E = 0 + \pi c \omega X^2$$

$$\Rightarrow \quad \boxed{\Delta E = \pi c \omega X^2}$$

4. Solution to the Undamped Free Vibration Differential Equations of Motion

In general, the differential equation of motion for one degree of freedom system is of the form:

$$\boxed{\begin{array}{l} \ddot{x} + \omega_n^2 x = 0 \\ \text{or} \quad \ddot{\theta} + \omega_\theta^2 \theta = 0 \end{array}}$$

General solution of the above differential equations is found by assuming Harmonic solution. As an example let's solve the differential equation

$$\ddot{x} + \omega_n^2 x = 0$$

$$\text{where } \omega_n = \sqrt{\frac{k}{m}}$$

Procedure:

① Assume harmonic solution

$$\boxed{x(t) = c\,e^{\lambda t}} \quad \text{Harmonic Solution}$$

② Substitute into differential equation

$$\ddot{x} = \frac{d^2x}{dt^2} = c\lambda^2 e^{\lambda t}$$

$$\rightarrow \ddot{x} + \omega_n^2 x = 0 \Rightarrow c\lambda^2 e^{\lambda t} + \omega_n^2 . c\,e^{\lambda t} = 0$$

$$\Rightarrow \quad c e^{\lambda t}(\lambda^2 + \omega_n^2) = 0$$

The above equation is satisfied iff,

- $c e^{\lambda t} = 0 \Rightarrow$ Trivial Solution
- $\lambda^2 + \omega_n^2 = 0$

$$\Rightarrow \lambda^2 = -\omega_n^2 \Rightarrow \boxed{\lambda = \pm i\omega_n}$$

The assumed harmonic solution becomes:

$$x(t) = c e^{\lambda t} \Rightarrow \boxed{x(t) = c_1 e^{-i\omega_n t} + c_2 e^{i\omega_n t}}$$

By using Euler's identities:

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

$$\begin{aligned} \hookrightarrow x(t) &= c_1 e^{-i\omega_n t} + c_2 e^{i\omega_n t} \\ &= c_1\left[\cos(\omega_n t) - i\sin(\omega_n t)\right] + c_2\left[\cos(\omega_n t) + i\sin(\omega_n t)\right] \\ &= (c_1 + c_2)\cos(\omega_n t) + i(c_2 - c_1)\sin(\omega_n t) \end{aligned}$$

$$\Rightarrow \boxed{x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)}$$

where $\begin{cases} A = c_1 + c_2 \\ B = i(c_2 - c_1) \end{cases}$

③ Determine the constants c_1 and c_2 OR A and B.

↳ The constants are determined from the initial conditions. Two conditions are necessary to evaluate the two constants uniquely.

(end of procedure...)

• The form $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$ is the most convential of the solution forms of the equation of motion $\ddot{x} + \omega_n^2 x = 0$. However, the properties of $x(t)$ are more easily by describing $x(t)$ under this form:

$$\boxed{x(t) = Amp \cdot \sin(\omega_n t - \phi)}$$

where $\begin{cases} Amp \equiv \text{amplitude}, & Amp = \sqrt{A^2 + B^2} \\ \phi \equiv \text{phase angle}, & \phi = -\tan^{-1}\left(\frac{A}{B}\right) \end{cases}$

<u>Proof:</u>

Consider,

$$\begin{cases} A = -Amp \cdot \sin(\phi) \\ B = Amp \cdot \cos(\phi) \end{cases}$$

Substituting into $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$

↳ $x(t) = -Amp\sin(\phi)\cos(\omega_n t) + Amp\cos(\phi)\sin(\omega_n t)$

Since, $\sin(u - v) = \sin u \cos v - \cos u \sin v$

↳ $\boxed{x(t) = Amp \cdot \sin(\omega_n t - \phi)}$ as prescribed

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The amplitude Amp and the phase angle are determined as follows:

- $A^2+B^2 = Amp^2\cos^2(\phi) + Amp^2\sin^2(\phi)$

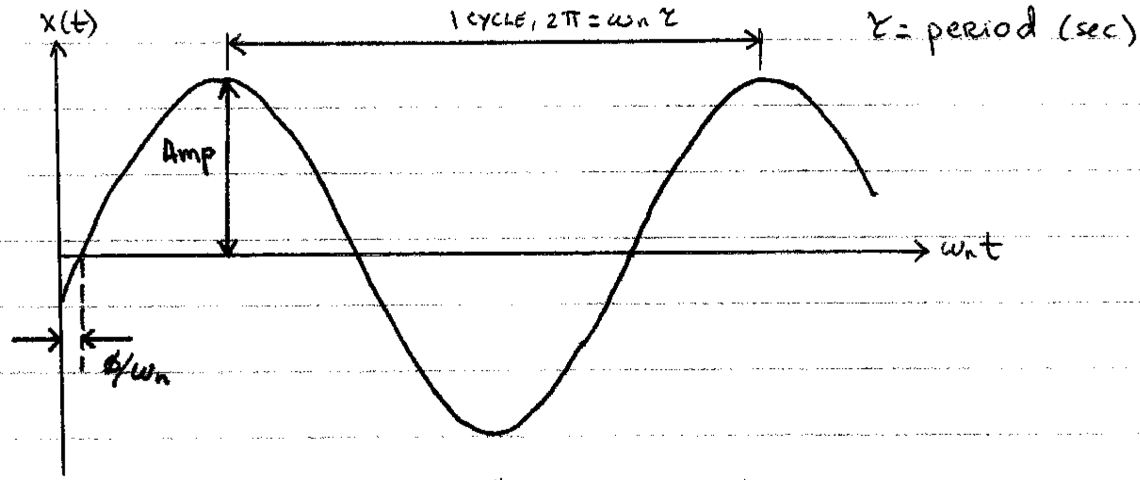
$\Rightarrow A^2+B^2 = Amp^2\,(\cancelto{1}{\cos^2(\phi)+\sin^2(\phi)})$

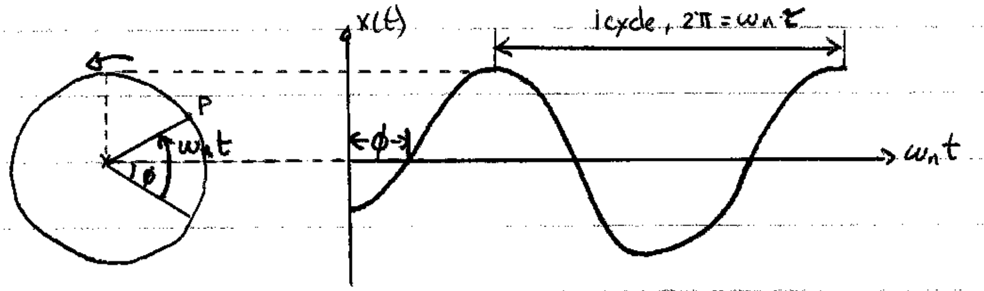
$\hookrightarrow \boxed{Amp = \sqrt{A^2+B^2}}$

- $\dfrac{A}{B} = -\dfrac{Amp\,\sin(\phi)}{Amp\,\cos(\phi)}$

$\Rightarrow \dfrac{A}{B} = -\tan(\phi)$

$\Rightarrow \boxed{\phi = -\tan^{-1}\left(\dfrac{A}{B}\right)}$





Point P makes one complete revolution (or cycle) during the time required for the angle $\omega_n t$ to increase by 2π. The time required is $\boxed{\tau = \dfrac{2\pi}{\omega_n}}$ Period (sec)

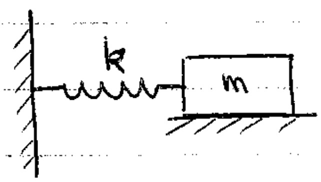
Since the period τ is the time required for one cycle, its inverse (frequency) is the number of cycles per unit time.

$$\boxed{f = \frac{1}{\tau}} \quad \text{frequency}, \ Hz = \frac{1}{sec}$$

$$\text{OR} \quad \boxed{f = \frac{\omega_n}{2\pi}} \quad \text{where } \omega_n \left(\frac{Rad}{sec}\right)$$

5. Energy Analysis for Undamped Free Vibrations

Consider the spring-mass system below:



(No damping)

- Equation of motion: $\ddot{x} + \omega_n^2 x = 0$, $\omega_n = \sqrt{\frac{k}{m}}$

The solution of the differential equation of motion is of the form $\boxed{x(t) = Amp \sin(\omega_n t - \phi)}$

$\hookrightarrow$ $v = \frac{dx}{dt} \Rightarrow \boxed{v = Amp\, \omega_n \cos(\omega_n t - \phi)}$

- Since there is no damping the energy is conserved,

$$\boxed{K + V = const}$$

where

$K \equiv$ kinetic energy.
$V \equiv$ potential energy.

$$\hookrightarrow \quad K + V = const$$

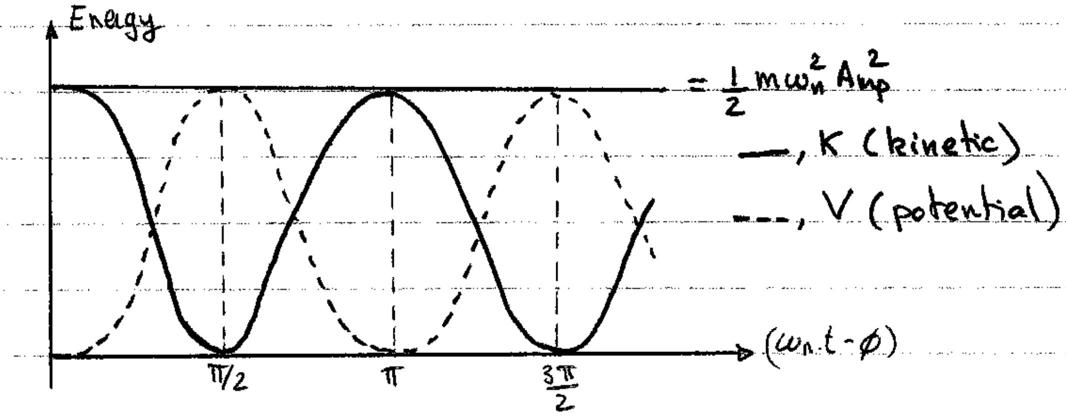
$$\Rightarrow \quad \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = const$$

$$k = \omega_n^2 . m$$

$$\Rightarrow \quad \frac{1}{2} m \, Amp^2 \omega_n^2 \cos^2(\omega_n t - \phi) + \frac{1}{2} k \, Amp^2 \sin^2(\omega_n t - \phi) = const$$

$$\Rightarrow \quad \frac{1}{2} m \omega_n^2 Amp^2 \left[\underbrace{\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi)}_{=1} \right] = const.$$

$$\hookrightarrow \quad \boxed{K_1 + V_1 = \frac{1}{2} m \omega_n^2 Amp^2 = const}$$



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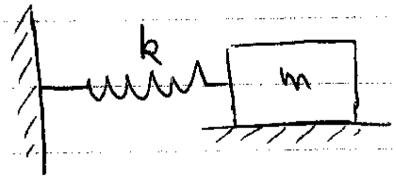
6. Phase-Plane or State-Space Representation

Consider the undamped spring-mass system



Equation of motion: $\ddot{x} + \omega_n^2 x = 0$, $\omega_n = \sqrt{\frac{k}{m}}$

Solution: $x(t) = Amp. \sin(\omega_n t - \phi)$

$$\Rightarrow \frac{x(t)}{Amp} = \sin(\omega_n t - \phi) \quad \text{—— (a)}$$

$$v = \frac{dx}{dt} = Amp. \omega_n \cos(\omega_n t - \phi)$$

$$\Rightarrow v = \dot{x} = Amp\, \omega_n \cos(\omega_n t - \phi)$$

$$\Rightarrow \frac{\dot{x}}{Amp\,\omega_n} = \cos(\omega_n t - \phi)$$

$$\text{OR} \quad \frac{\dot{x}}{Amp.\omega_n} = \frac{y}{Amp} = \cos(\omega_n t - \phi) \quad \text{—— (b)}$$

where $y = \frac{\dot{x}}{\omega_n}$

Adding and squaring (a) and (b),

$$\frac{x^2}{Amp^2} + \frac{y^2}{Amp^2} = \overbrace{\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi)}^{1}$$

$$\Rightarrow \boxed{\frac{x^2}{Amp^2} + \frac{y^2}{Amp^2} = 1}$$

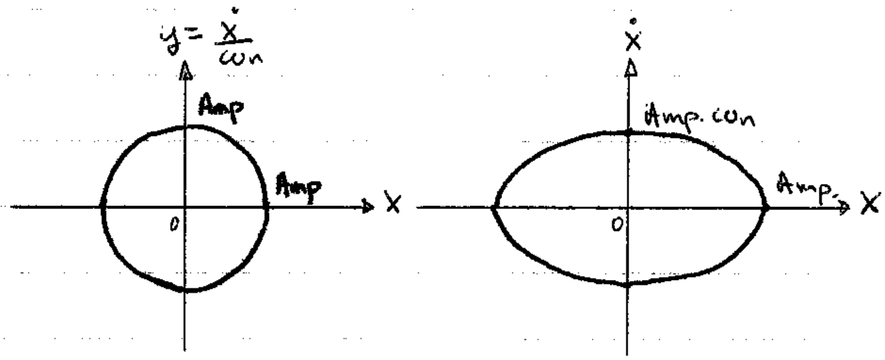
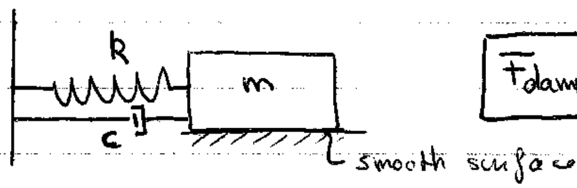


Fig. Phase-Plane of Undamped System.

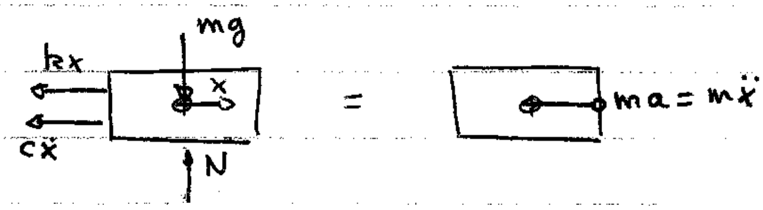
7. Free-Vibration with Viscous Damping

Consider the system shown below,



$$\boxed{F_{damping} = -c\,v = -c\dot{x}}$$

① FBD



② $\Sigma \bar{F} = m\bar{a}$

$$\Sigma F_x = m a_x \Rightarrow \quad -kx - c\dot{x} = m\ddot{x}$$

$$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + kx = 0} \quad \text{— (a)}$$

To solve the differential equation of motion let assume harmonic solution

(continues next page...)

Harmonic solution: $\boxed{x(t) = \bar{c}\, e^{-\lambda t}}$ — (b)

$$\begin{cases} x = \bar{c}\, e^{-\lambda t} \\ \dot{x} = -\lambda \bar{c}\, e^{-\lambda t} \\ \ddot{x} = \lambda^2 \bar{c}\, e^{-\lambda t} \end{cases}$$

The differential equation of motion can be rewritten as:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\Rightarrow \quad e^{-\lambda t}\bar{c}\left(\lambda^2 m - \lambda c + k\right) = 0 \quad \text{— (c)}$$

The above equation is satisfied iff:

- $\bar{c}\, e^{-\lambda t} = 0 \Rightarrow$ TRIVIAL Solution
- $m\lambda^2 - c\lambda + k = 0$
  the roots of which are:

$$\lambda_{1,2} = \frac{c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\Rightarrow \quad \boxed{\lambda_{1,2} = \frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}} \quad \text{— (d)}$$

$\hookrightarrow \quad x(t) = \bar{c}\, e^{-\lambda t}$

$$\Rightarrow \quad x(t) = \bar{c}_1\, e^{-\lambda_1 t} + \bar{c}_2\, e^{-\lambda_2 t}$$

$$\Rightarrow \quad \boxed{x(t) = \bar{c}_1\, e^{-\left\{\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + \bar{c}_2\, e^{-\left\{\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t}}$$

$\hookrightarrow$ (e)

where $\bar{c}_1$ and $\bar{c}_2$ are arbitrary constants to be determined from the initial conditions

- <u>Critical Damping Constant, c_c:</u>

The critical damping c_c is defined as the value of the damping constant c for which the radical in (d) becomes zero.

$$\boxed{c_c = c}\,, \qquad \left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\Rightarrow \quad c_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n$$

↳ $\boxed{c_c = 2m\omega_n}$ Critical Damping Constant

- <u>Damping Ratio, ξ:</u>

For any damped system, the damping ratio ξ is defined as:

$$\boxed{\xi = \frac{c}{c_c}}$$

↳ $$\frac{c}{c_c} = \frac{c}{2m\omega_n} \Rightarrow \xi = \frac{c}{2m\omega_n}$$

$$\text{or} \quad \boxed{\frac{c}{2m} = \xi\omega_n} \quad - (f)$$

Substituting $\frac{c}{2m} = \xi \omega_n$ into the general solution of $x(t)$ (e)

↳ $$\boxed{x(t) = \bar{c}_1 e^{-(\xi + \sqrt{\xi^2 - 1})\omega_n t} + \bar{c}_2 e^{-(\xi - \sqrt{\xi^2 - 1})\omega_n t}} \quad \text{— (h)}$$

- The behavior of the solution $x(t)$ depends upon the magnitude of the damping (or the roots λ_1 and λ_2). It can be observed that for $\xi = 0$, the general solution reduces to the undamped solution derived in p. 25-26

- If $\xi \neq 0$, three cases are possible

<u>Case 1</u>: Critically Damped, $\xi = 1$

<u>Case 2</u>: Underdamped, $\xi < 1$

<u>Case 3</u>: Overdamped, $\xi > 1$

- For the above 3 cases the following initial conditions are considered:

$$\boxed{\begin{aligned} x(t)\big|_{t=0} &= x_0 \\ v(t)\big|_{t=0} = \dot{x}(t)\big|_{t=0} &= \dot{x}_0 \end{aligned}} \quad \text{Initial Conditions}$$

Some Notation:

$$\lambda_{1,2} = \frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Using $\frac{c}{2m} = \xi \omega_n$ and $\omega_n^2 = \frac{k}{m}$

$\hookrightarrow \lambda_{1,2} = \xi \cdot \omega_n \pm \sqrt{\xi^2 \omega_n^2 - \omega_n^2}$

$$\Rightarrow \boxed{\lambda_{1,2} = \omega_n \left\{ \xi \pm \sqrt{\xi^2 - 1} \right\}}$$

Case 1: Critically Damped $\xi = 1$:

$\xi = 1 \Rightarrow \lambda = \lambda_{1,2}$ (Repeated roots)

$\hookrightarrow x(t) = \bar{c}_1 e^{-\omega_n t} + \bar{c}_2 e^{-\omega_n t}$

$x(t) = e^{-\omega_n t}(\bar{c}_1 + \bar{c}_2 t)$ (For repeated roots)

Applying the two initial conditions to determine $\bar{c}_1$ & $\bar{c}_2$:

$x(t)\big|_{t=0} = x_0$, $\boxed{\bar{c}_1 = x_0}$

$\dot{x}(t)\big|_{t=0} = \dot{x}_0$, $\left(-\omega_n e^{-\omega_n t} \bar{c}_1 - \omega_n e^{-\omega_n t} \bar{c}_2 t + e^{-\omega_n t} \bar{c}_2\right|_{t=0} = \dot{x}_0$

$$\Rightarrow -\omega_n \cancel{\bar{c}_1}^{x_0} - 0 + \bar{c}_2 = \dot{x}_0$$

$$\Rightarrow \boxed{\bar{c}_2 = \dot{x}_0 + \omega_n x_0}$$

Substituting $\bar{c}_1$ and $\bar{c}_2$ into the general solution

$$x(t) = e^{-\omega_n t}(\bar{c}_1 + \bar{c}_2 t)$$

$$\Rightarrow \boxed{x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right]e^{-\omega_n t}}$$

Critically Damped solution

* $\lim\limits_{x \to \infty} x(t) \approx \lim\limits_{x \to \infty} e^{-\omega_n t} = 0$, the solution diminishes to zero as x grows.

Case 2: Underdamped $\xi < 1$

$\xi < 1 \Rightarrow \xi^2 - 1 < 0 \qquad \sqrt{-1} = i$

$$\Rightarrow \begin{cases} \lambda_1 = (\xi + i\sqrt{1-\xi^2})\omega_n \\ \lambda_2 = (\xi - i\sqrt{1-\xi^2})\omega_n \end{cases}$$

The general solution $x(t)$ becomes:

$$x(t) = \bar{c}_1 e^{-(\xi + i\sqrt{1-\xi^2})\omega_n t} + \bar{c}_2 e^{-(\xi - i\sqrt{1-\xi^2})\omega_n t}$$

$$= e^{-\xi\omega_n t}\left\{\bar{c}_1 e^{-i\sqrt{1-\xi^2}\omega_n t} + \bar{c}_2 e^{i\sqrt{1-\xi^2}\omega_n t}\right\}$$

Using Euler's identities

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

$$\hookrightarrow \quad x(t) = e^{-\xi\omega_n t}\left\{(\bar{c}_1+\bar{c}_2)\cos\left(\sqrt{1-\xi^2}\,\omega_n t\right) + i\,(\bar{c}_2-\bar{c}_1)\sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)\right\}$$

$$\Rightarrow \quad x(t) = e^{-\xi\omega_n t}\left\{ \bar{A}\cos\left(\sqrt{1-\xi^2}\,\omega_n t\right) + \bar{B}\sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)\right\}$$

where $\begin{cases} \bar{A} = \bar{c}_1+\bar{c}_2 \\ \bar{B} = i(\bar{c}_2-\bar{c}_1) \end{cases}$

- Following the same procedure as in p: 27

$$\begin{cases} \bar{A} = -\text{Amp.}\sin\phi \\ \bar{B} = \text{Amp}\cos\phi \end{cases}$$

- Substituting into $x(t)$ and using the trigonometric identity

$$\sin(u-v) = \sin u\cos v - \cos u \cdot \sin v$$

$$\hookrightarrow \quad \boxed{x(t) = \text{Amp.}\, e^{-\xi\omega_n t}\sin\left(\sqrt{1-\xi^2}\,\omega_n t - \phi\right)}$$

In order to determine the parameters,

$$\begin{cases} \text{Amp} = \sqrt{\bar{A}^2+\bar{B}^2} \\ \phi = -\tan^{-1}\left(\dfrac{\bar{A}}{\bar{B}}\right) \end{cases}$$

the two arbitrary constants $\bar{c}_1$ and $\bar{c}_2$ must be determined from the initial conditions

$$\begin{cases} x(t)\big|_{t=0} = x_0 \\ \dot{x}(t)\big|_{t=0} = \dot{x}_0 \end{cases}$$

By applying the initial conditions, the time response can be rewritten under the following forms **(see attached Maple file)**

$$\begin{cases} \bar{A} = x_0 \\ \bar{B} = \dfrac{\dot{x}_0 + \xi \omega_n x_0}{\sqrt{1-\xi^2}\,\omega_n} \end{cases};$$

$$\boxed{x(t) = e^{-\xi\omega_n t}\left\{ x_0 \cos\left(\sqrt{1-\xi^2}\,\omega_n t\right) + \frac{\dot{x}_0 + \xi\omega_n x_0}{\sqrt{1-\xi^2}\,\omega_n} \sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)\right\}}$$

OR

$$\boxed{x(t) = Amp\; e^{-\xi\omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t - \phi\right)}$$

where

$$Amp = \sqrt{\bar{A}^2 + \bar{B}^2}$$

$$\phi = -\tan^{-1}\left(\frac{\bar{A}}{\bar{B}}\right)$$

(continues next page...)

$x := c1 \cdot exp\left(-\left(\zeta + sqrt\left(1-\zeta^2\right)\right) \cdot omega \cdot t\right) + c2 \cdot exp\left(-\left(\zeta - sqrt\left(1-\zeta^2\right)\right) \cdot \omega \cdot t\right);$

$$c1\, e^{\left(-\left(\zeta+\sqrt{1-\zeta^2}\right)\omega t\right)} + c2\, e^{\left(-\left(\zeta-\sqrt{1-\zeta^2}\right)\omega t\right)} \quad (1)$$

$dx := diff(x, t);$

$$-c1\left(\zeta+\sqrt{1-\zeta^2}\right)\omega\, e^{\left(-\left(\zeta+\sqrt{1-\zeta^2}\right)\omega t\right)} - c2\left(\zeta-\sqrt{1-\zeta^2}\right)\omega\, e^{\left(-\left(\zeta-\sqrt{1-\zeta^2}\right)\omega t\right)} \quad (2)$$

$t := 0;$

$$0 \quad (3)$$

$eq1 := x;$

$$c1 + c2 \quad (4)$$

$eq2 := dx;$

$$-c1\left(\zeta+\sqrt{1-\zeta^2}\right)\omega - c2\left(\zeta-\sqrt{1-\zeta^2}\right)\omega \quad (5)$$

$solve(\{eq1 - xo = 0, eq2 - vo = 0\}, \{c1, c2\}):$

(6)

$c1 := \frac{1}{2}\frac{\sqrt{1-\zeta^2}\left(\omega\zeta xo - \omega\sqrt{1-\zeta^2}\, xo + vo\right)}{\omega\left(-1+\zeta^2\right)};$

$$\frac{1}{2}\frac{\sqrt{1-\zeta^2}\left(\omega\zeta xo - \omega\sqrt{1-\zeta^2}\, xo + vo\right)}{\omega\left(-1+\zeta^2\right)} \quad (7)$$

$c2 := -\frac{1}{2}\frac{\left(\omega\zeta xo + \omega\sqrt{1-\zeta^2}\, xo + vo\right)\sqrt{1-\zeta^2}}{\omega\left(-1+\zeta^2\right)};$

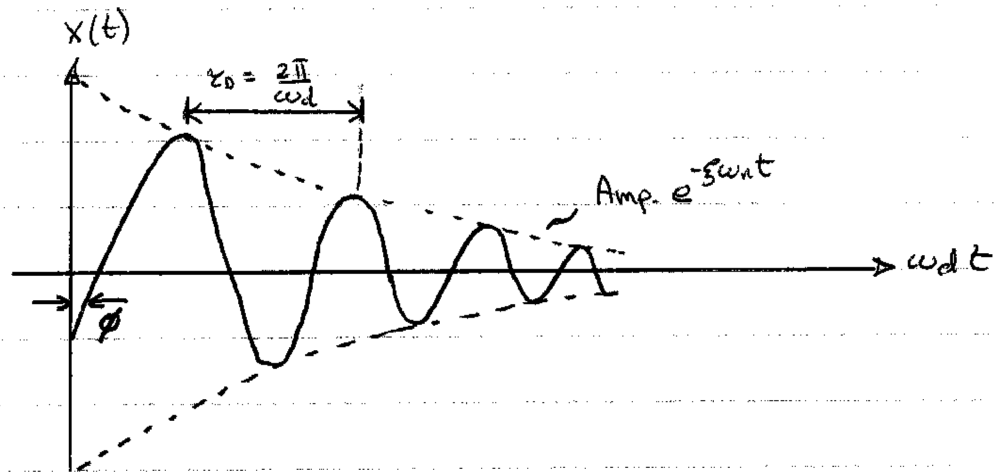
$$-\frac{1}{2}\frac{\left(\omega\zeta xo + \omega\sqrt{1-\zeta^2}\, xo + vo\right)\sqrt{1-\zeta^2}}{\omega\left(-1+\zeta^2\right)} \quad (8)$$

$A := simplify(c1 + c2);$

$$xo \quad (9)$$

$B := simplify(I \cdot (c2 - c1));$

$$\frac{\mathrm{I}\,(\omega\zeta xo + vo)}{\sqrt{1-\zeta^2}\,\omega} \quad (10)$$



The quantity $\boxed{\omega_d = \sqrt{1-\xi^2}\,\omega_n}$ is called the damped frequency.

<u>Case 3:</u> Overdamped $\xi > 1$

$$\xi > 1 \Rightarrow \xi^2 - 1 > 0$$

$$\Rightarrow \begin{cases} \lambda_1 = \left(\xi + \sqrt{\xi^2 - 1}\right)\omega_n \\ \lambda_2 = \left(\xi - \sqrt{\xi^2 - 1}\right)\omega_n \end{cases}$$

The general solution x(t) becomes:

$$x(t) = \bar{c}_1 \, e^{-\left(\xi + \sqrt{\xi^2-1}\right)\omega_n t} + \bar{c}_2 \, e^{-\left(\xi - \sqrt{\xi^2-1}\right)\omega_n t}$$

Applying the two initial conditions to determine $\bar{c}_1$ and c_2

(continues next page...)

ME-400

$$x(t)\Big|_{t=0} = x_0$$

$$\hookrightarrow \quad \boxed{\bar{c}_1 + \bar{c}_2 = x_0} \quad -(i)$$

$$\dot{x}(t)\Big|_{t=0} = \dot{x}_0$$

$$\hookrightarrow \quad \frac{dx}{dt} = \dot{x} = -\bar{c}_1\left(\xi + \sqrt{\xi^2-1}\right)\omega_n e^{-(\xi+\sqrt{\xi^2-1})\omega_n t} - \bar{c}_2\left(\xi - \sqrt{\xi^2-1}\right) e^{-(\xi-\sqrt{1-\xi^2})\omega_n t}$$

$$\Rightarrow \quad \boxed{-\bar{c}_1\left(\xi + \sqrt{\xi^2-1}\right)\omega_n - \bar{c}_2\left(\xi - \sqrt{\xi^2-1}\right)\omega_n = \dot{x}_0} \quad -(ii)$$

Solving the two equations (i) and (ii) for $\bar{c}_1$ and $\bar{c}_2$:

$$\boxed{\begin{aligned} \bar{c}_1 &= -\frac{x_0\omega_n\left(\xi - \sqrt{\xi^2-1}\right) + \dot{x}_0}{2\omega_n\sqrt{\xi^2-1}} \\ \bar{c}_2 &= \frac{x_0\omega_n\left(\xi + \sqrt{\xi^2-1}\right) + \dot{x}_0}{2\omega_n\sqrt{\xi^2-1}} \end{aligned}}$$

The general solution for the overdamped case is obtained by substituting $\bar{c}_1$ and $\bar{c}_2$ into:

$$\boxed{x(t) = \bar{c}_1 e^{-(\xi+\sqrt{\xi^2-1})\omega_n t} + \bar{c}_2 e^{-(\xi-\sqrt{\xi^2-1})\omega_n t}}$$

It could be shown that the response is non-periodic and that the response diminishes exponentially.

(continues next page...)

Summary of Results:

Critically-Damped Systems

When $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = 0$ (equivalent $\zeta = 1$ or $c = c_c$), the characteristic equation has repeated real roots. The displacement solution for this kind of system is,

$$x(t) = (c_1 + c_2 t)e^{-\omega_n t}$$
$$\Rightarrow x(t) = e^{-\omega_n t}\left[x_0 + (v_0 + \omega_n x_0)t\right]$$

The critical damping factor c_c can be interpreted as the *minimum damping* that results in non-periodic motion (i.e. simple decay).

Underdamped Systems

When $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} < 0$ (equivalent to $\zeta < 1$ or $c < c_c$), the characteristic equation has a pair of complex conjugate roots. The displacement solution for this kind of system is,

$$x(t) = c_1 e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n t} + c_2 e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n t}$$
$$= e^{-\zeta\omega_n t}\left[d_1 \cos(\omega_d t) + d_2 \sin(\omega_d t)\right]$$
$$\Rightarrow x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{Exponentially decay}} \underbrace{\left[x_0 \cos(\omega_d t) + \frac{v_0 + \zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\right]}_{\text{Periodic motion}}$$

An alternate but equivalent solution is given by,

$$x(t) = A_0 \underbrace{e^{-\zeta\omega_n t}}_{\text{Exponentially decay}} \underbrace{\cos(\omega_d t - \varphi_0)}_{\text{Periodic}}$$

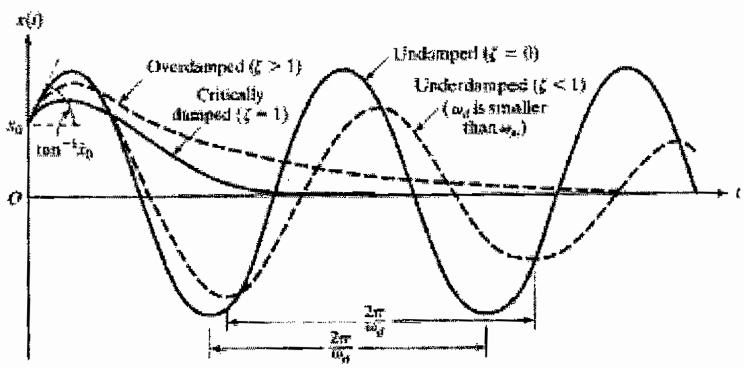
Overdamped Systems

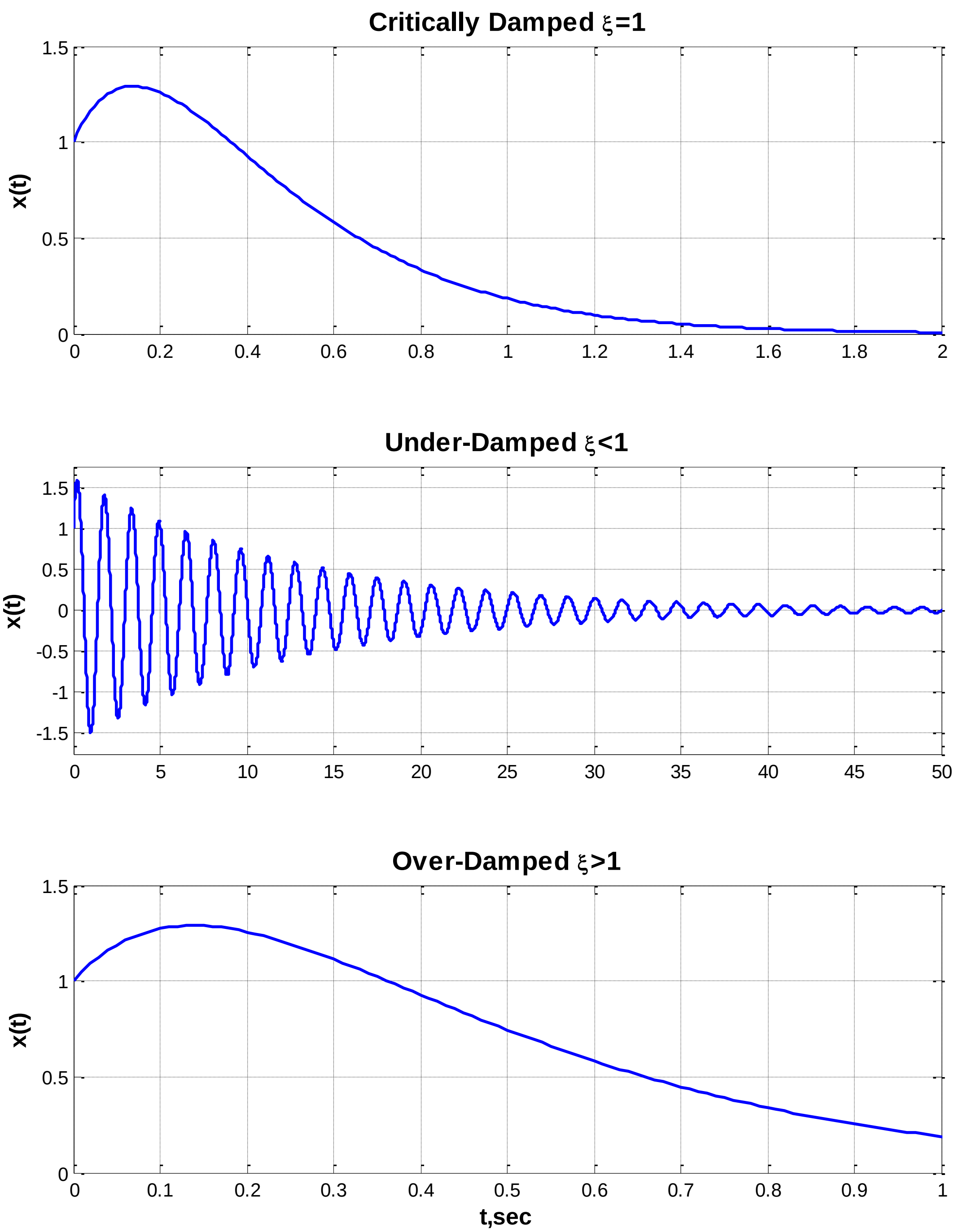
When $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} > 0$ (equivalent $\zeta > 1$ or $c > c_c$), the characteristic equation has two distinct real roots. The displacement solution for this kind of system is,

$$x(t) = c_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + c_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

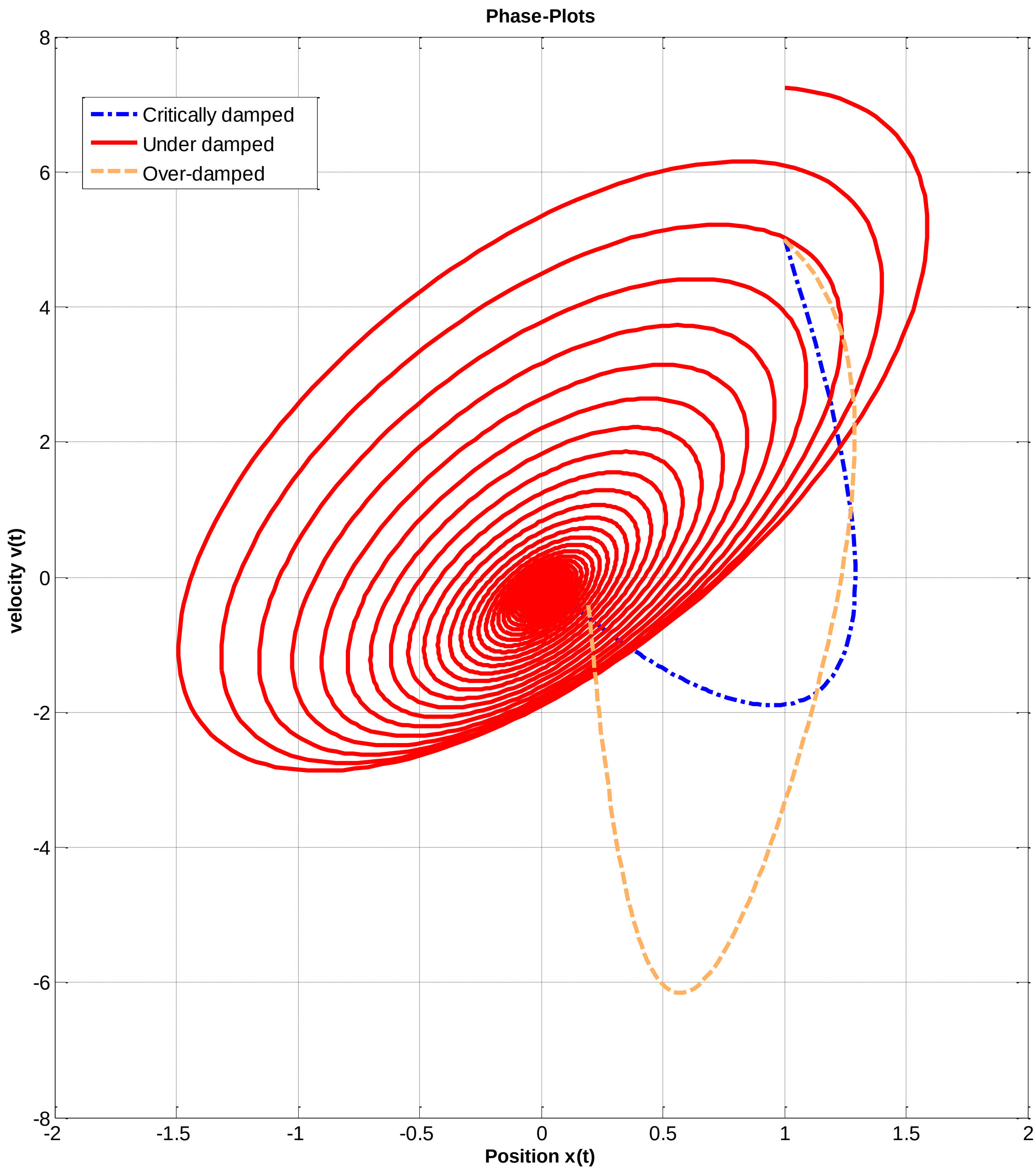
$$\Rightarrow x(t) = \frac{x_0\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)+v_0}{2\omega_n\sqrt{\zeta^2-1}} e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + \frac{-x_0\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)-v_0}{2\omega_n\sqrt{\zeta^2-1}} e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Figure Comparison of Motion with Different damping characteristics





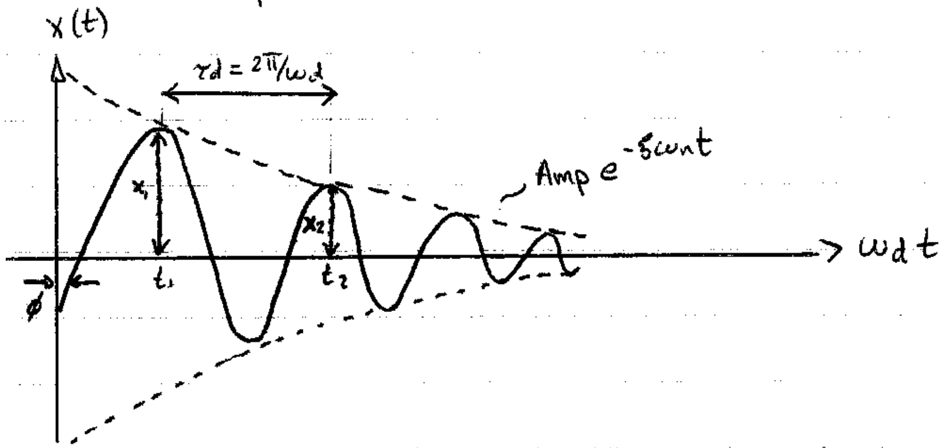
Critically Damped ξ=1
x(t)
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Under-Damped ξ<1
x(t)
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
40
45
50
Over-Damped ξ>1
x(t)
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t,sec



Phase-Plots
Critically damped
Under damped
Over-damped
velocity v(t)
Position x(t)
8
6
4
2
0
-2
-4
-6
-8
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Logarithmic Decrement (only for under-damped case)

The logarithmic decrement represents the rate at which the amplitude of the free-damped vibration decreases. It is defined as the natural logarith of the ratio of any two successive amplitudes.



Recalling the solution for the under-damped case (p. 39)

$$x(t) = Amp \cdot e^{-\xi\omega_n t} \sin(\sqrt{1-\xi^2}\,\omega_n t - \phi)$$

$$\text{or} \quad x(t) = Amp\, e^{-\xi\omega_n t} \sin(\omega_d t - \phi)$$

$$\rightarrow \quad \frac{x_1}{x_2} = \frac{x(t=t_1)}{x(t=t_2)} = \frac{Amp \cdot e^{-\xi\omega_n t_1} \sin(\omega_d t_1 - \phi)}{Amp \cdot e^{-\xi\omega_n t_2} \sin(\omega_d t_2 - \phi)}$$

or $t_2 = t_1 + \tau_d$ where $\tau_D = 2\pi/\omega_d$

$$\sin(\omega_d t_2 - \phi) = \sin(2\pi + \omega_d t_1 - \phi) = \sin(\omega_d t_1 - \phi)$$

$$\rightarrow \quad \frac{x_1}{x_2} = \frac{e^{-\xi\omega_n t_1}}{e^{-\xi\omega_n t_2}} = \frac{e^{-\xi\omega_n t_1}}{e^{-\xi\omega_n (t_1+\tau_d)}} = e^{\xi\omega_n \tau_d} \quad \text{— (i)}$$

The logarithmic decrement can be obtained from (i)

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \xi \omega_n \tau_d = \xi \frac{\omega_n 2\pi}{\omega_d} = \frac{2\pi\xi \cancel{\omega_n}}{\sqrt{1-\xi^2}\cdot\cancel{\omega_n}}$$

$$\Rightarrow \boxed{\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

Hence, to determine the amount of damping, it suffices to measure any two consecutive displacements x_1 and x_2 one cycle apart ($\delta = \ln\left(\frac{x_1}{x_2}\right)$), and obtain ξ from

$$\boxed{\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}}$$

For small damping,

$$\xi \approx \frac{\delta}{2\pi} \Rightarrow \boxed{\delta = 2\pi\xi}$$

- The damping factor ξ can also be determined by measuring two displacements separated by any number of complete cycles

$$\frac{x_1}{x_{j+1}} = \frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot\frac{x_3}{x_4}\cdots\frac{x_j}{x_{j+1}} = e^{(\xi\omega_n T)j} = e^{j\xi\omega_n T}$$

$$\rightarrow \boxed{\delta = \frac{1}{j}\ln\frac{x_1}{x_{j+1}}}$$

(continues next page...)

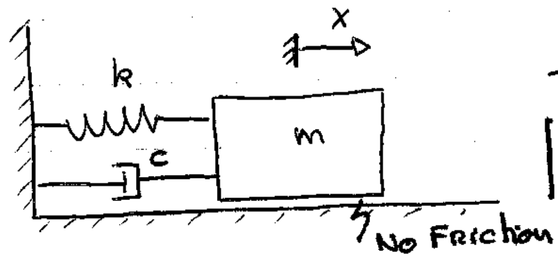
Exercice 7

For the system below

$m = 200$ kg

$k = 1000$ N/m

$c = 100 \frac{N}{m}.sec.$



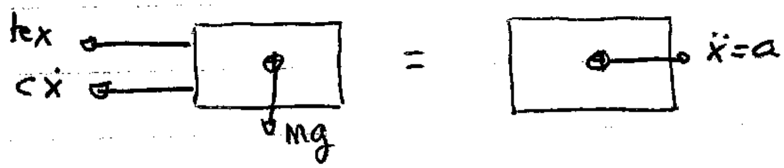
Viscous Damper

$x_0 = 5.0$

$\dot{x}_0 = 1.0$

Determine the equation of motion and solve it a) analytically and b) numerically.

- FBD:



- Newton's Equations (Motion ⇒ translation)

$$\Sigma F_x = m a_x \Rightarrow -kx - c\dot{x} = m\ddot{x}$$

$$\hookrightarrow \boxed{m\ddot{x} + c\dot{x} + kx = 0} \text{ Eqt. of Motion.}$$

Critical damping

$$c_c = 2m\omega_n = 2m\sqrt{\frac{k}{m}}$$

$$\Rightarrow c_c = 2(200)\sqrt{\frac{1000}{200}}$$

$$\Rightarrow \boxed{c_c = 894.4272}$$

Damping Ratio

$$\xi = \frac{c}{c_c} = \frac{100}{894.4272}$$

$$\Rightarrow \boxed{\xi = 0.1118}$$

$\hookrightarrow$ $\xi < 1 \Rightarrow$ Under-Damped

Analytical Solution:

$$x(t) = e^{-\xi \omega_n t}\left[x_0 \cos(\omega_d t) + \frac{\dot{x}_0 + \xi \omega_n x_0}{\omega_d} \sin(\omega_d t) \right]$$

where $\omega_d = \sqrt{1-\xi^2}\,\omega_n$ (Damped Frequency)

(see Matlab file ...)

Numerical Solution:

The differential equation of motion $m\ddot{x} + c\dot{x} + kx = 0$ is solved numerically using the Runge-Kutta method in MATLAB (ode function).

Assume $y = \dot{x}$, $z = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ $x \to$ initial displacement, $\dot{x} \to$ initial velocity

$$\hookrightarrow \quad z = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} \Rightarrow \dot{z} = \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} y \\ \ddot{x} \end{Bmatrix}$$

From the equation of motion $m\ddot{x} + c\dot{x} + kx = 0$

$$\hookrightarrow \quad \boxed{\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x}$$

Then, the assume function $\dot{z}$ becomes,

$$\dot{z} = \begin{Bmatrix} y \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} \dot{x} \\ -\frac{c}{m}\dot{x} - \frac{k}{m}x \end{Bmatrix} = \begin{Bmatrix} z(2) \\ -\frac{c}{m}z(2) - \frac{k}{m}z(1) \end{Bmatrix}$$

The above equation is what is needed by the MATLAB function ODE to solve the equation of motion numerically.

For more details on how to program it in MATLAB see attached MATLAB input files

Results:

Plots of $x(t)$ vs t
$\dot{x}(t)$ vs t
$x(t)$ vs $\dot{x}(t)$
show that the responses obtained analytically or numerically are identical.

MATLAB INPUT CODES

Analytical Solution:

```
% ME-400
% Free Vibration for Viscous Damping.
clear all;
clc;
%-----------------------------------------------------------------------
----
%This program solves the ODE of the Free-Vibration-Viscoulsy-Damped
% system analytically.
%-----------------------------------------------------------------------
----
%Input Data------------------------------------------------------------
----
m=200;              % mass
k=1000;             % stiffness
c=100;              % damping

% set initial conditions
xo = 5.0;
vo = 1.0;

dt=0.05;  % integration time step size
n=400;
tmax=dt*n;
t=[0:dt:tmax];
%=======================================================================
====
%Main Program
%ANALYTICAL SOLUTION

% Response characteristics
omega=sqrt(k/m);
cc=2*m*omega;
ci=c/cc;

if ci==1;
    disp('Critically Damped System...')
    gamma1=omega*ci;
    gamma2=omega*ci;
    c1=xo;
    c2=vo+omega*xo;
    x=(c1+c2*t)*exp(-gamma1*t);
    dx=-gamma1*exp(-gamma1*t)*(c1+c2*t)+c2*exp(-gamma1*t);

elseif ci>1;
    disp('Overdapmed Damped System...')
    gamma1=omega(ci+sqrt(ci^2-1));
    gamma2=omega(ci-sqrt(ci^2-1));
    c1=-(xo*omega*(ci-sqrt(ci^2-1)+vo))/(2*omega*sqrt(ci^2-1));
    c2=(xo*omega*(ci-sqrt(ci^2-1)+vo))/(2*omega*sqrt(ci^2-1));
    x=c1.exp(-gamma1*t)+c2.exp(-gamma2*t);
    dx=-gamma1*c1*exp(-gamma1*t)-gamma2*c2*exp(-gamma2*t);
```

```
else ci<1;
    disp('Underdamped Damped System...')
    %gamma1=omega(ci+i*sqrt(1-ci^2));
    %gamma2=omega(ci-i*sqrt(1-ci^2));
    %c1 = 1/2*sqrt(1-ci^2)*(omega*ci*xo-omega*sqrt(1-
ci^2)*xo+vo)/(omega*(-1+ci^2));
    %c2 = -1/2*(omega*ci*xo+omega*sqrt(1-ci^2)*xo+vo)*sqrt(1-
ci^2)/(omega*(-1+ci^2));
    %x=c1.exp(-gamma1*t)+c2.exp(-gamma2*t);
    %dx=-gamma1*c1*exp(-gamma1*t)-gamma2*c2*exp(-gamma2*t);
    % or
    omega_d=(1-ci^2)^(1/2)*omega;
    c1=xo;
    c2=(vo+ci*omega*xo)/(1-ci^2)^(1/2)/omega;
    x=exp(-ci*omega*t).*(c1*cos((1-ci^2)^(1/2)*omega*t)...
        +c2*sin((1-ci^2)^(1/2)*omega*t));
    dx=-ci*omega*exp(-
ci*omega*t).*(c1*cos(omega_d*t)+c2*sin(omega_d*t))...
        +exp(-ci*omega*t).*(-
c1*sin(omega_d*t)*omega_d+c2*cos(omega_d*t)*omega_d);
end

z(:,1)=x;
z(:,2)=dx;

figure(1);
subplot(1,3,1)
plot(t,z(:,1))
xlabel('time')
ylabel('displacement')
grid on

% plot velocity result
%figure(2);
subplot(1,3,2)
plot(t,z(:,2))
title([' ANALYTICAL SOLUTION   ','\zeta =' num2str(ci)])
xlabel('time')
ylabel('velocity')
grid

% Phase plot displacement Vs. velocity
%figure(2);
subplot(1,3,3)
plot(z(:,1),z(:,2))
xlabel('displacement')
ylabel('velocity')
grid
```

(end of analytical solution)

MATLAB INPUT CODES

Numerical Solution:

Main Program: Free_Vib_Viscous.m

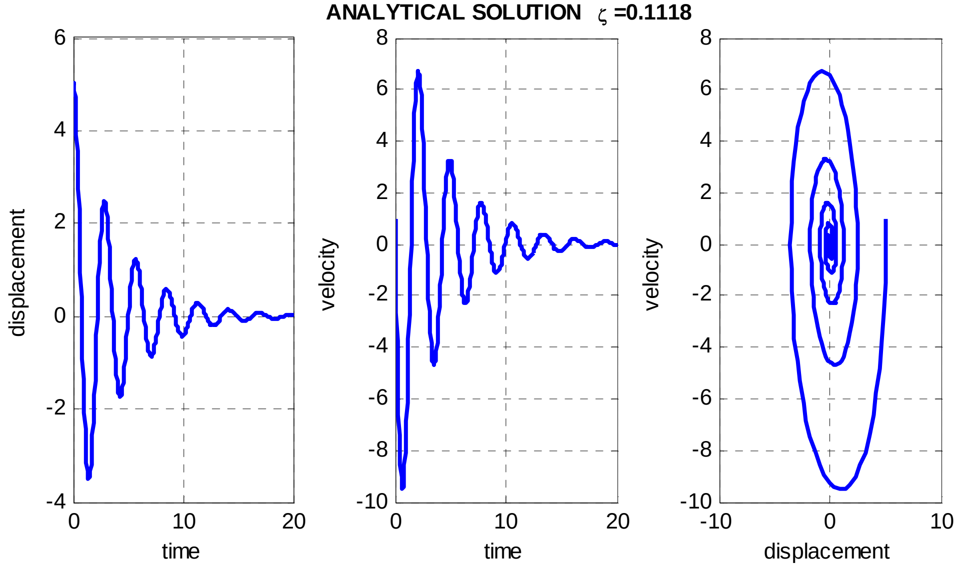
```
% ME-400
% Free Vibration for Viscous Damping.
clear all;
clc;
%-----------------------------------------------------------------------
----
%This program solves the ODE of the Free-Vibration-Viscoulsy-Damped
% system numerically.
%-----------------------------------------------------------------------
----
%Input Data------------------------------------------------------------
----
m=200;              % mass
k=1000;             % stiffness
c=100;              % damping
% mass, stiffness, damping ==> dfunc1.m  !!!!!!!IMPORTANT!!!!!!!!
% set itegration parameters
dt=0.05;  % integration time step size
n=400;
tmax=dt*n;
tspan=[0:dt:tmax];  % integration time interval from t=0~tmax

% set initial conditions
xo = 5.0;
vo = 1.0;
z0=[xo,vo];

%=======================================================================
====
%Main Program
%NUMERIACL SOLUTION
% Solve ODE (eqt. of motion) using Runge-Kutta method
[t,z]=ode45('dfunc1', tspan, z0);

% results are returned in an array z, where
% z(:,1) is the displacement vector
% z(:,2) is the first order time derivative of disp, i.e., velocity
% plot displacement result

% Response characteristics
omega=sqrt(k/m);
cc=2*m*omega;
ci=c/cc;

if ci==1;
    disp('Critically Damped System...')
elseif ci>1;
    disp('Overdapmed Damped System...')
```

```
else ci<1;
    disp('Underdamped Damped System...')
end

figure(1);
subplot(1,3,1)
plot(t,z(:,1))
xlabel('time')
ylabel('displacement')
grid on

% plot velocity result
%figure(2);
subplot(1,3,2)
plot(t,z(:,2))
title([' RUNGE-KUTTA NUMERICAL INTERGRATION SOLUTION   ','\zeta ='
num2str(ci)])
xlabel('time')
ylabel('velocity')
grid

% Phase plot displacement Vs. velocity
%figure(2);
subplot(1,3,3)
plot(z(:,1),z(:,2))
xlabel('displacement')
ylabel('velocity')
grid
```

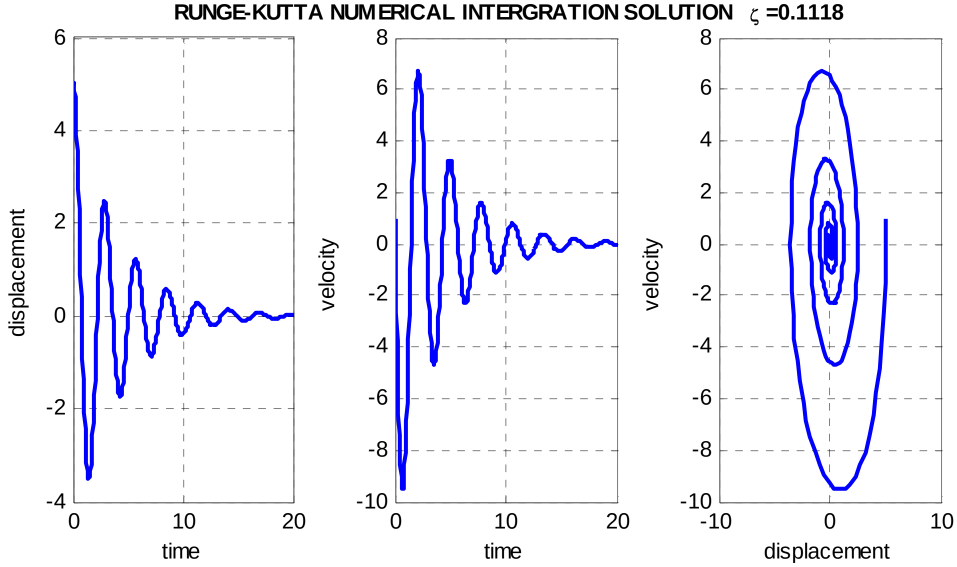
Function dfunc1.m

```
% dfunc1.m
function f = dfunc1(t,z)

%Input Data------------------------------------------------------------
----
m=200;              % mass
k=1000;             % stiffness
c=100;              % damping
%=====================================================================
===

f=zeros(2,1);
f(1)=z(2);
f(2)=-c/m*z(2)-k/m*z(1);

%return

% Response Characteristic
omega=sqrt(k/m);
cc=2*m*omega;
ci=c/cc;
```

ANALYTICAL SOLUTION ζ =0.1118



RUNGE-KUTTA NUMERICAL INTERGRATION SOLUTION ζ =0.1118

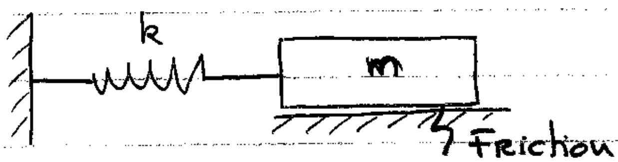


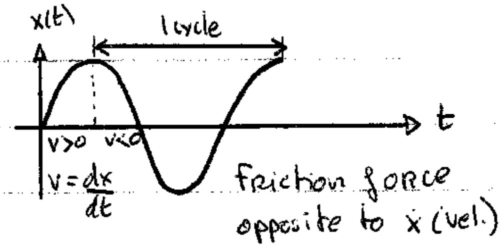
8. Free Vibration with Coulomb Damping

- In many mechanical systems, Coulomb or Dry-friction dampers are used because of their simplicity and convenience. As stated in section 3.2 Coulomb damping arises when bodies slide on dry surfaces, and the damping force equals:

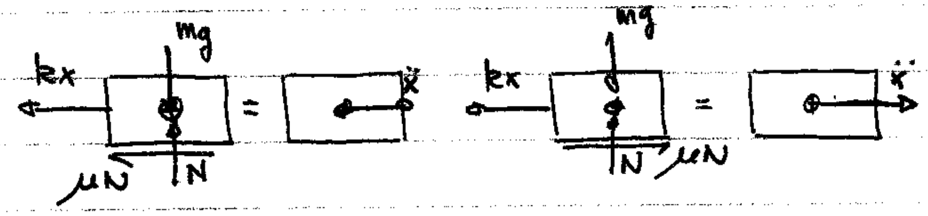
$$\boxed{F = \mu N = \mu m g}$$

- Consider a single degree of freedom system with dry-friction as shown in the figure below. Since the friction force varies with the direction of the motion.





FBD



Case 1: Case 2:

Once put into motion the block oscillates back and forth around the equilibrium position (which changes in time).

Case 1: $m\ddot{x} = -kx - \mu N \Rightarrow \boxed{m\ddot{x} + kx = -\mu N}$ — (i)

Case 2: $m\ddot{x} = -kx + \mu N \Rightarrow \boxed{m\ddot{x} + kx = \mu N}$ — (ii)

Equation (i) and (ii) can be expressed as a single equation (using $N = mg$)

↳ $m\ddot{x} + \mu mg + kx = 0$ Case 1

$m\ddot{x} - \mu mg + kx = 0$ Case 2

↳ $$\boxed{m\ddot{x} + \mu mg \, \mathrm{sgn}(\dot{x}) + kx = 0}$$

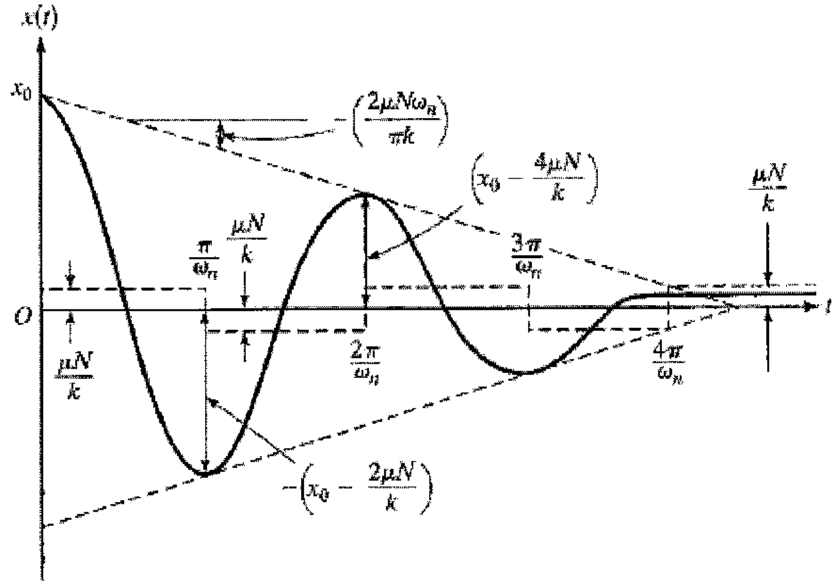
where $\mathrm{sgn}(\dot{x})$ is called the signum function.

$$\begin{cases} \mathrm{sgn}(\dot{x}) = +1 & \text{if } \dot{x} > 0 \\ \mathrm{sgn}(\dot{x}) = -1 & \text{if } \dot{x} < 0 \end{cases}$$

The above equation of motion is nonlinear and a simple analytical solution does not exist, numerical methods can be used to solve it conveniently. However, before solving the differential equation of motion numerically, let us give some characteristics of a system with Coulomb damping

(see next page →

Response with Coulomb Damping



1. The equation of motion is nonlinear with Coulomb damping, while it is linear with viscous damping

2. The natural frequency $\omega_n = \sqrt{k/m}$ of the system is unaltered with the addition of the Coulomb damping, while it is reduced $\omega_d = \sqrt{(1-\xi^2)}\omega_n$ (damped frequency) with the addition of viscous damping

3. The motion is periodic, the object in the system is vibrating back and forth around an equilibrium point

4. In each successive cycle the amplitude of the motion is reduced by $4\mu mg/k$ so the amplitudes at the end of any two consecutive cycles ($2\pi / \omega_n$) are related by

$$X_m = X_{m-1} - \frac{4\mu mg}{k}$$

5. The amplitude reduces linearly with Coulomb damping, while it reduces exponentially wit viscous damping

$$-\frac{2\mu mg\omega_n}{\pi k}$$

6. The position where the object stops, or its equilibrium position, could potentially be at a completely different position than when initially at rest because the system is nonlinear.

Exercice 8:

Find the free vibration response of a spring-mass system subjected to Coulomb damping with the following data:

$m = 10\ kg$ $\qquad x(0) = 5\ m$

$k = 200\ N/m$ $\qquad \dot{x}(0) = 0\ m/sec$

$\mu = 0.5$

↳ See Matlab input file + Results (Plots)

↳ Equation of Motion:

$$m\ddot{x} + \mu m g\, \mathrm{sign}(\dot{x}) + kx = 0$$

Assume $y = \dot{x}$, $z = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$

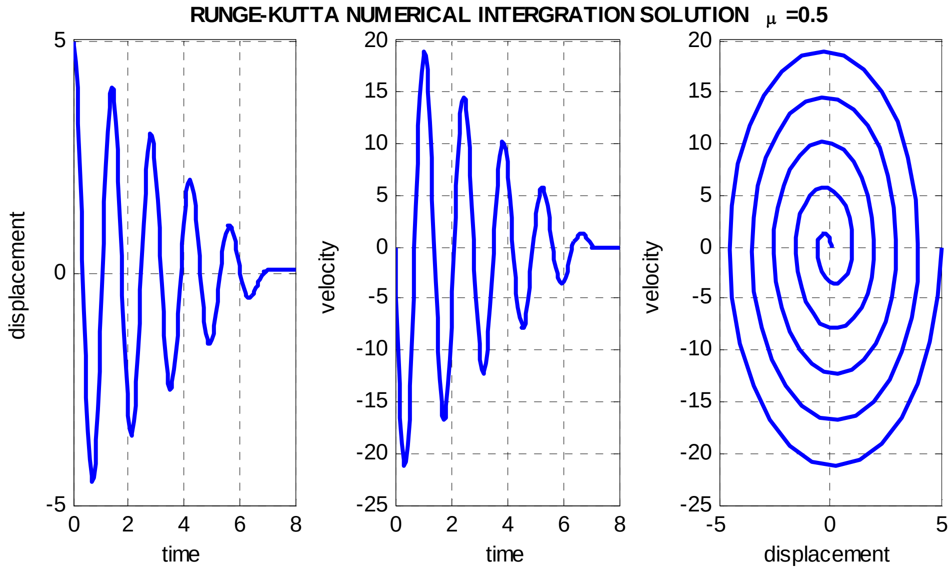
$$\hookrightarrow \quad \dot{z} = \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ \ddot{x} \end{Bmatrix}$$

From equation of motion: $\ddot{x} = -\frac{k}{m}x - \mu g\, \mathrm{sign}(\dot{x})$

$$\hookrightarrow \quad \dot{z} = \begin{Bmatrix} y \\ \ddot{x} \end{Bmatrix} = \begin{Bmatrix} z(2) \\ -\frac{k}{m} z(1) - \mu g\, \mathrm{sign}(z(2)) \end{Bmatrix}$$

sign ≡ Sign function in Matlab

Coulomb Damping

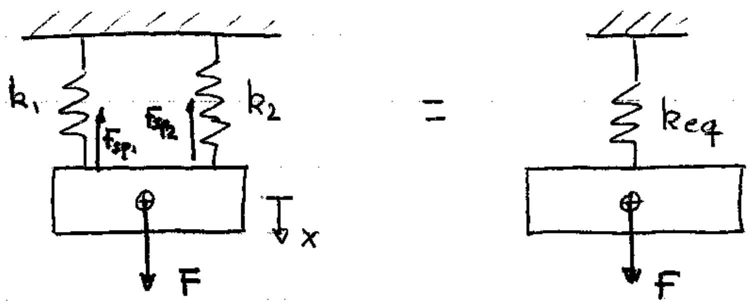



```
% ME-400
% Free Vibration for Coulomb Damping.
clear all;
clc;
%-----------------------------------------------------------------------
----
%This program solves the ODE of the Free-Vibration-Coulomb-Damped
% system numerically.
%-----------------------------------------------------------------------
----
%Input Data-------------------------------------------------------------
----
m=10;                   % mass
k=200;                  % stiffness
mu=0.5;                 % dry damping coefficient
% mass, stiffness, damping ==> dfunc1.m  !!!!!!!IMPORTANT!!!!!!!!
% set itegration parameters
dt=0.05;  % integration time step size
n=160;
tmax=dt*n;
tspan=[0:dt:tmax];  % integration time interval from t=0~tmax

% set initial conditions
xo = 5.0;
vo = 0.0;
z0=[xo;vo];

%=======================================================================
====
%Main Program
%NUMERIACL SOLUTION
% Solve ODE (eqt. of motion) using Runge-Kutta method
disp('Numerical Integration...')
[t,z]=ode23('dfunc1', tspan, z0);
```

```
figure(1);
subplot(1,3,1)
plot(t,z(:,1))
xlabel('time')
ylabel('displacement')
grid on

% plot velocity result
%figure(2);
subplot(1,3,2)
plot(t,z(:,2))
title([' RUNGE-KUTTA NUMERICAL INTERGRATION SOLUTION   ','\mu ='
num2str(mu)])
xlabel('time')
ylabel('velocity')
grid

% Phase plot displacement Vs. velocity
%figure(2);
subplot(1,3,3)
plot(z(:,1),z(:,2))
xlabel('displacement')
ylabel('velocity')
grid



% dfunc1.m
function f = dfunc1(t,z)

%Input Data-------------------------------------------------------------
----
m=10;                   % mass
k=200;                  % stiffness
mu=0.5;                 % dry-damping coefficient
g=9.81;                 % gravitational acceleration
%==============================================================
===

f=zeros(2,1);
f(1)=z(2);
f(2)=-mu*g*sign(z(2))-k/m*z(1);

%continue
```

9. Equivalent Springs

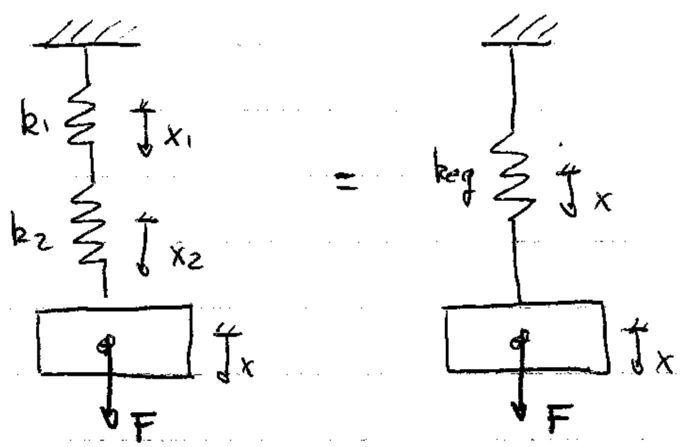
9.1 Springs in Parallel



$$F = F_{sp1} + F_{sp2} = k_1 x + k_2 x = \underbrace{(k_1 + k_2)}_{k_{eq}} x$$

$$\hookrightarrow \boxed{k_{eq} = \sum_{n=1}^{\infty} k_n}$$

9.2 Springs in Series



$$x = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = F \underbrace{\left(\frac{1}{k_1} + \frac{1}{k_2} \right)}_{1/k_{eq}}$$

$$\hookrightarrow \boxed{\frac{1}{k_{eq}} = \sum_{n=1}^{\infty} \frac{1}{k_n}}$$

10. Extended Lagrangian's Equation

For a system subjected to non-conservative forces ($\bar{F} = -\bar{\nabla} V$) φ and subjected to damping, the classic Lagrange's equation (pages 6-7) becomes

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial \mathcal{D}}{\partial \dot{x}} = \varphi}$$

where

$\mathcal{D} \equiv$ dissipation function

(continues next page)

Exercice 9 :

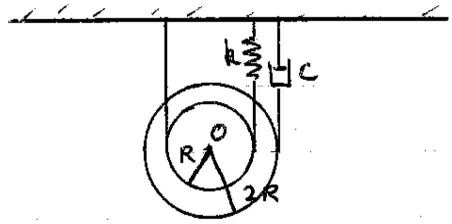
$R = 0.8\ m$

$m = 20\ kg$

$I = \frac{mR^2}{2}$

$k = 60\ N/m$

$c = 24\ \frac{N\text{-}sec^2}{m}$



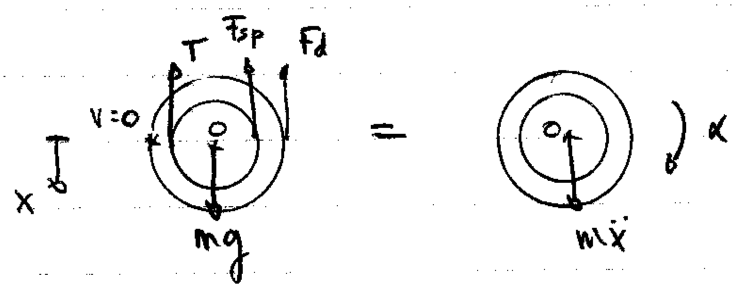
Determine the equation of motion in function of the linear acceleration, velocity, and displacement. Assume that

$$\left\| \begin{array}{l} x(0) = 0 \\ v = \frac{dx}{dt} = \dot{x} \Big|_{t=0} = 0 \end{array} \right.$$

Types of motion $\Rightarrow$ Translation & Rotation

$\Sigma \bar{F} = m\bar{a}$ $\quad\quad$ $\Sigma M = I\alpha$

FBD



$F_{sp} = 2kx$

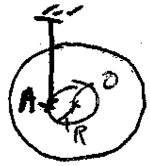
$F_d = 3c\dot{x}$

Equations of motion

$$\Sigma \bar{F} = m\bar{a} \Rightarrow mg - T - 2kx - 3c\dot{x} = m\ddot{x} \quad \text{— (i)}$$

$$\Sigma M_o = I_o\alpha \Rightarrow T(R) - 2kx(R) - 3c\dot{x}(2R) = \frac{mR^2}{2}\alpha \quad \text{— (ii)}$$

* <u>Kinematics</u>



$V_A = 0 \qquad \omega = 0$ (at rest at $t=0$)

0 since $\omega = 0$

$$\cancel{\bar{a}_o}^{=\ddot{x}} = \cancel{\bar{a}_A}^{0} + \bar{\alpha} \times \bar{r}_{o/A} - \cancel{\omega^2 \bar{r}_{o/A}}^{0}$$

$$\Rightarrow \begin{pmatrix} \ddot{x} \\ 0 \\ 0 \end{pmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -\alpha \\ 0 & R & 0 \end{vmatrix} \Rightarrow \ddot{x} = R\alpha \text{ OR } \boxed{\alpha = \frac{\ddot{x}}{R}}$$

Rewritting (i) and (ii),

$$-T = m\ddot{x} + 2kx + 3c\dot{x} - mg$$

$$T = \ddot{x}\left(\frac{m}{2}\right) + 2kX + 6c\dot{x}$$

(i)+(ii)

$$0 = \ddot{x}\left(m + \frac{m}{2}\right) + 4kx + 9c\dot{x} - mg$$

$$\Rightarrow \boxed{\frac{3m}{2}\ddot{x} + 4kx + 9c\dot{x} = mg}$$

OR

$$\boxed{\ddot{x} + \frac{8kx}{3m} + \frac{6c}{m}\dot{x} = \frac{2}{3}g} \quad -(\ast)$$

where

$\omega_n = \sqrt{\frac{8k}{3m}}$ is the natural frequency.

The equation of motion $(\ast)$ is non-homogeneous (meaning that right-hand-side term is non equal to zero).
In order to make it homogeneous the following change of variable is necessary

$$\boxed{x = \bar{x} + x_{equ}}$$

where

At equilibrium $\ddot{x} = \dot{x} = 0$

$$\Rightarrow \quad \frac{8k}{\cancel{3}m} x_{equ} = \frac{2}{\cancel{3}} g \quad \Rightarrow \quad \boxed{x_{equ} = \frac{mg}{4k}}$$

Then $x = \bar{x} + x_{eq} = \bar{x} + \dfrac{mg}{4k}$

Substituting into (*)

$$\Rightarrow \quad \ddot{\bar{x}} + \frac{6c}{m}\dot{\bar{x}} + \frac{8k}{3m}\left(\bar{x} + \frac{m\cancel{g}}{\cancel{4k}}\right) = \cancel{\frac{2}{3}g}$$

$$\Rightarrow \quad \boxed{\ddot{\bar{x}} + \frac{6c}{m}\dot{\bar{x}} + \frac{8k}{3m}\bar{x} = 0} \quad (**)$$

The above equation is homogeneous, and it can be solved using standard procedures.

① Assume $\bar{x}(t) = de^{-\lambda t}$

② Substitute into (**)

$$\Rightarrow \quad de^{-\lambda t}\left[\lambda^2 - \frac{6c}{m}\lambda + \frac{8k}{3m}\right] = 0$$

The above equation is satisfied iff,

- $de^{-\lambda t} = 0 \Rightarrow$ Trivial solution

- $\lambda^2 - \dfrac{6c}{m}\lambda + \dfrac{8k}{3m} = 0$

Solving the quadratic equation,

$$\lambda^2 - \frac{6c}{m}\lambda + \frac{8k}{3m} = 0$$

$$\lambda_{1,2} = \frac{\frac{6c}{m} \pm \sqrt{\left(\frac{6c}{m}\right)^2 - 4\left(\frac{8k}{3m}\right)}}{2}$$

The critical damping is defined as the value of c for which $\left(\frac{6c_c}{2m}\right)^2 - 4\left(\frac{8k}{3m}\right) = 0$

$$\Rightarrow \frac{36c_c^2}{4m^2} = \frac{32k}{3m}$$

$$\Rightarrow c_c = \frac{4 \times 32km}{36 \times 3} \Rightarrow \boxed{c_c = \frac{8}{9}m}$$

Numerical application:

$$c_c = \frac{8}{9}(20) = 17.78$$

Recalling the damping ratio definition $\boxed{\xi = \frac{c}{c_c}}$

$$\hookrightarrow \xi = \frac{24}{17.78} \Rightarrow \xi \simeq 1.35 > 1 \Rightarrow \text{Over-Damped}$$

$$\lambda_{1,2} = \frac{\frac{6c}{m} \pm \sqrt{\left(\frac{6c}{m}\right)^2 - 4\left(\frac{8k}{3m}\right)}}{2}$$

$$\hookrightarrow \begin{cases} \lambda_1 = 5.8271 \\ \lambda_2 = 1.3729 \end{cases}$$

The solution to the homogeneous problem becomes

$$\boxed{\bar{x}(t) = d e^{-\lambda_i t} = d_1 e^{-5.8271 t} + d_2 e^{-1.3729 t}}$$

and the general solution for $x(t)$ is

$$\boxed{x(t) = \bar{x} + x_{eq} = d_1 e^{-5.8271 t} + d_2 e^{-1.3729 t} + \frac{mg}{4k}}$$

Applying the following boundary conditions,

$$\begin{cases} x(t=0) = 0 \\ v(t=0) = 0 \end{cases}$$

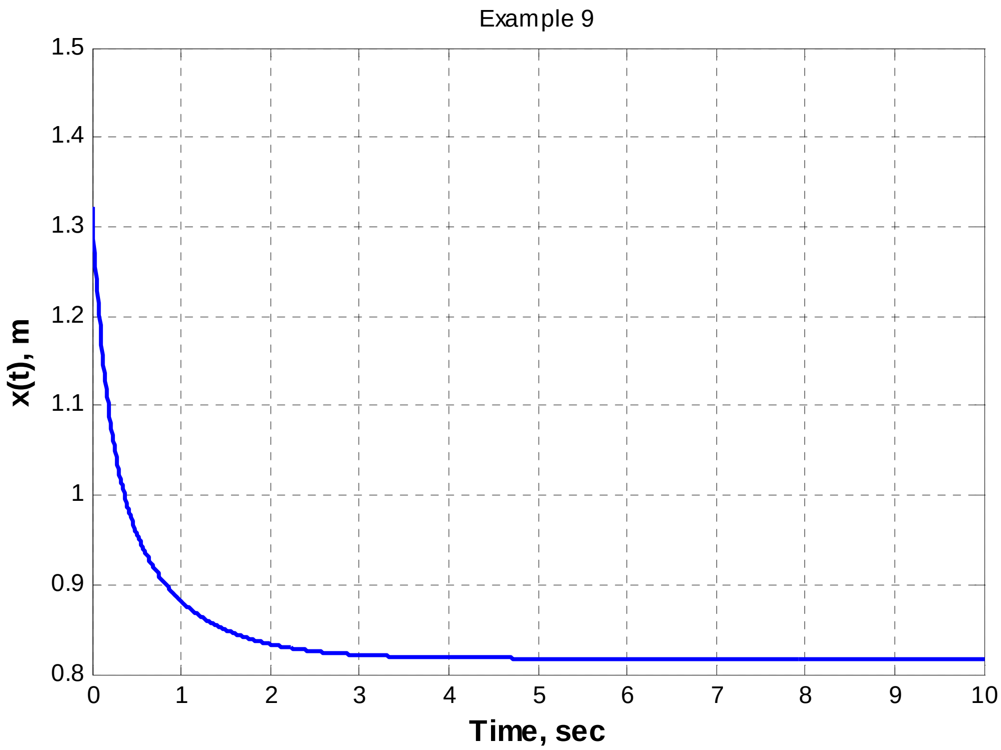
$$\rightarrow \begin{cases} d_1 + d_2 = -\frac{mg}{4k} \\ -5.8271\, d_1 - 1.3729\, d_2 = 0 \end{cases}$$

Solving for d_1 and d_2

$$\rightarrow \boxed{\begin{aligned} d_1 &= 0.2520 \\ d_2 &= -1.0695 \end{aligned}}$$

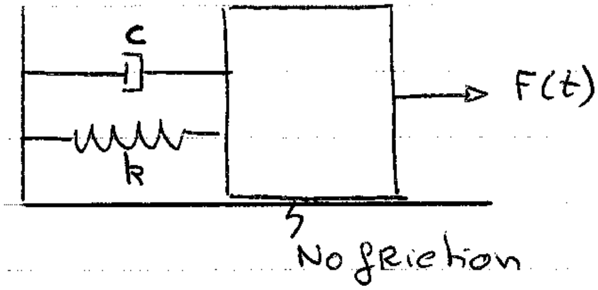
Substituting into $x(t) = d_1 e^{-5.8271 t} + d_2 e^{-1.3729 t} + \frac{mg}{4k}$, and plotting for $t = 1-10$ sec (see next page)

Example 9

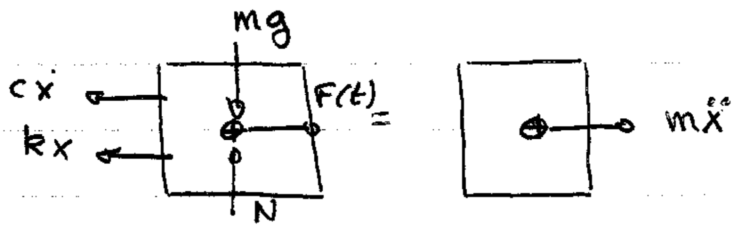


11. Damped Forced Vibration

Consider the following system,



Equation of motion :



Motion $\Rightarrow$ Translation

$$\Sigma \bar{F} = m\bar{a} \Rightarrow \quad F(t) - kx - c\dot{x} = m\ddot{x}$$

$$\text{OR} \quad \boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

where the natural frequency of the system is

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

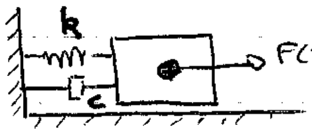
In the next pages, the equation of motion would be solved for $F(t) = F_0 e^{i\Omega t}$, based excitation, and to a general force.

11.1 Response to Harmonic Forcing Function, $F(t) = F_0 e^{i\Omega t}$

In this section the exponential notation would be selected over the trigonometric for the sake of simplicity.

For a system subjected to an harmonic forcing function, the equation of motion becomes

$$\boxed{m\ddot{x} + c\dot{x} + kx = F_0 e^{i\Omega t}} \quad (i)$$



$F(t) = F_0 e^{i\Omega t}$

Consider only the real part of $e^{i\Omega t}$

$Re(e^{i\Omega t}) = \cos(\Omega t)$

Assume a solution in the form

$$\boxed{x(t) = X_0 e^{i\Omega t}} - (ii)$$

Substituting (ii into (i),

$$\rightarrow \quad X_0 \cancel{e^{i\Omega t}} \left[-m\Omega^2 + i(c\Omega) + k\right] = F_0 \cancel{e^{i\Omega t}}$$

$$\Rightarrow \quad \boxed{X_0 = \frac{F_0}{(k - m\Omega^2) + i(c\Omega)} = F_0 H(\Omega)}$$

where

$$\boxed{H(\Omega) = \frac{1}{(k - m\Omega^2) + i(c\Omega)}}$$

The expression for X_0 is complex meaning that there is a phase difference between the excitation force $F(t)$ and the displacement $x(t)$.

ME-400

After some algebra,

$$X_0 = \frac{F_0}{(k-m\Omega^2)+i(c\Omega)}$$

$(a+ib)(a-ib) = a^2+b^2$

$$\Rightarrow X_0 = \frac{F_0\left[(k-m\Omega^2)-i(c\Omega)\right]}{\left[(k-m\Omega^2)+i(c\Omega)\right]\left[(k-m\Omega^2)-i(c\Omega)\right]}$$

$$\Rightarrow X_0 = \frac{F_0}{(k-m\Omega^2)^2+(c\Omega)^2}\left[(k-m\Omega^2)-i(c\Omega)\right]$$

- Now, recalling from complex algebra that:

$\bar{z} = a+ib = |\bar{z}|e^{i\phi}$; $\bar{z}^* = a-ib = |\bar{z}^*|e^{-i\phi}$

$|\bar{z}| = \text{magnitude} = \sqrt{(a+ib)(a-ib)} = \sqrt{\bar{x}\cdot\bar{x}^*} = \sqrt{a^2+b^2}$

$\phi = \text{Phase} = \tan^{-1}\left(\frac{b}{a}\right)$

↳ The complex quantity $(k-m\Omega^2)-i(c\Omega)$ can be rewritten as

Amplitude term is square root

$$(k-m\Omega^2)-i(c\Omega) = \underbrace{\left[(k-m\Omega^2)^2+(c\Omega)^2\right]}_{\text{Amplitude}} e^{-i\phi}$$

where $\phi = \tan^{-1}\left(\dfrac{-c\Omega}{(k-m\Omega^2)}\right)$

(continues next page...)

The expression for x_o becomes,

$$\hookrightarrow \quad X_o = \frac{F_o}{[(k-m\Omega^2)^2+(c\Omega)^2]}\left[(k-m\Omega^2)-i(c\Omega)\right]$$

$$\Rightarrow \quad X_o = \frac{F_o.[(k-m\Omega^2)^2+(c\Omega)^2]^{1/2}e^{-i\phi}}{[(k-m\Omega^2)^2+(c\Omega)^2]}$$

$$\Rightarrow \quad X_o = \frac{F_o}{[(k-m\Omega^2)+(c\Omega)^2]^{1/2}}\,e^{-i\phi}$$

Substituting into the assumed harmonic solution,

$$x(t) = X_o e^{i\Omega t}$$

$$\Rightarrow \boxed{x(t) = \frac{F_o}{[(k-m\Omega^2)^2+(c\Omega)^2]^{1/2}}\,e^{i(\Omega t-\phi)}}$$

- With some additional algebraic manipulations the above expression can be rewritten as,

$$(k-m\Omega^2)^2+(c\Omega)^2$$
$$= m^2\left\{\left(\frac{k}{m}-\Omega^2\right)^2+\left(\frac{c}{m}\Omega\right)^2\right\} \quad \text{using } \omega_n^2=\frac{k}{m} \text{ \& } \frac{c}{m}=2\xi\omega_n$$
$$= m^2\left\{(\omega_n^2-\Omega^2)^2+(2\xi\omega_n\Omega)^2\right\}$$
$$= m^2\omega_n^4\left\{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right\}$$

(continues next page...)

The solution $x(t) = \dfrac{F_o}{\left[(k-m\Omega^2)^2+(c\Omega)^2\right]^{1/2}} e^{i(\Omega t-\phi)}$

$$\Rightarrow x(t) = \frac{F_o}{m} \frac{1}{\omega_n^2\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}} e^{i(\Omega t-\phi)}$$

$$\Rightarrow \boxed{x(t) = \frac{F_o}{m} H_m(\Omega)\, e^{i(\Omega t-\phi)}}$$

where

$$\begin{cases} H_m(\Omega) = \dfrac{1}{\omega_n^2\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}} \\[2em] \phi = -\tan^{-1}\left[\dfrac{2\xi\left(\frac{\Omega}{\omega_n}\right)}{1-\left(\frac{\Omega}{\omega_n}\right)^2}\right] \end{cases}$$

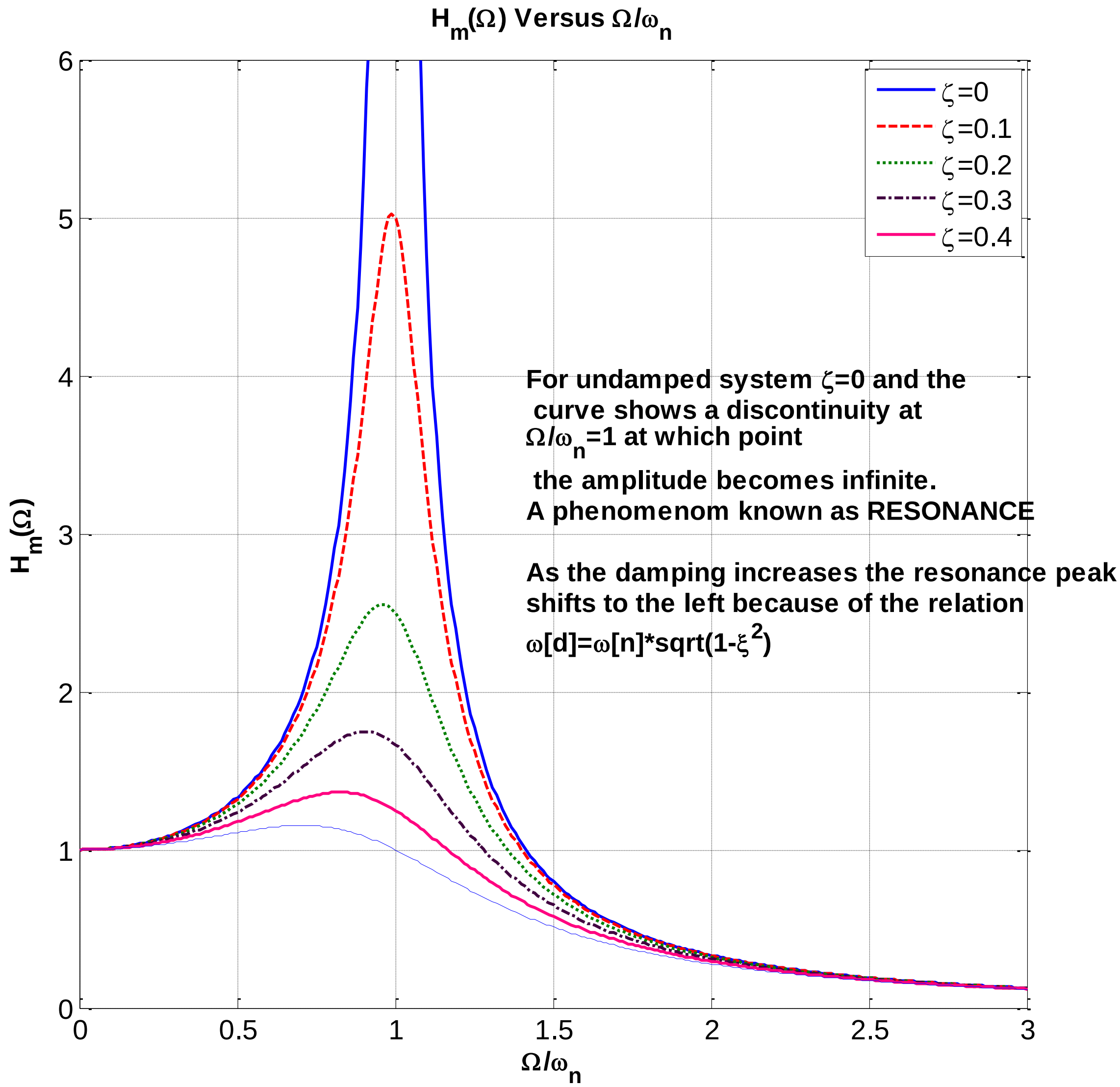
* Comments for figure of $H_m(\Omega)$ Vs $\left(\frac{\Omega}{\omega_n}\right)$

1- The damping tends to diminish the amplitude
2- shifts the peaks to the left at the maximum obtained at Resonance $\left(\frac{\Omega}{\omega_n}\right) = 1$

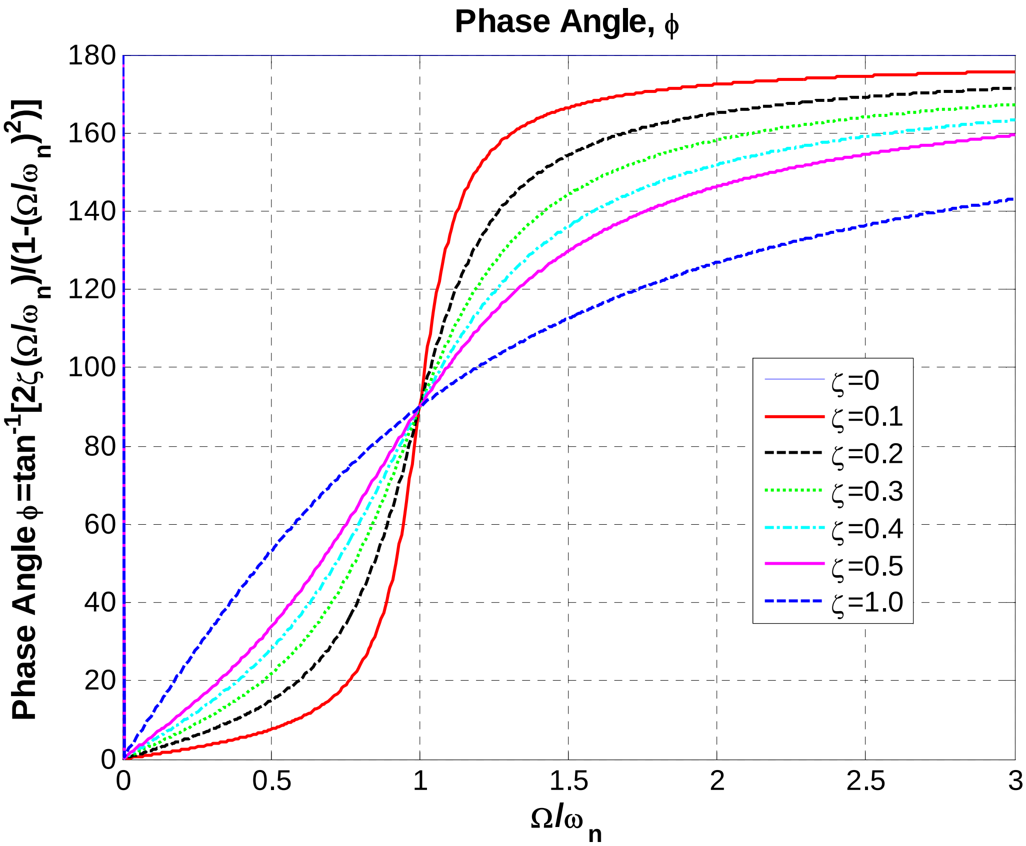
* To find the value at which the peaks occur, the extremum of $H_m(\Omega)$ is considered.

$$\frac{dH_m(\Omega)}{d\Omega} = 0 \Rightarrow \Omega = \omega_n\sqrt{1-2\xi^2}$$

OR $\boxed{\left(\frac{\Omega}{\omega_n}\right) = \sqrt{1-2\xi^2}}$

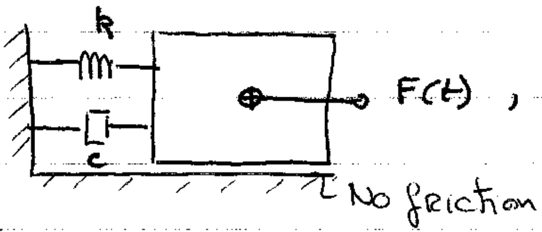


$H_m(\Omega)$ Versus Ω/ω_n
ζ=0
ζ=0.1
ζ=0.2
ζ=0.3
ζ=0.4
For undamped system ζ=0 and the
curve shows a discontinuity at
Ω/ω_n=1 at which point
the amplitude becomes infinite.
A phenomenom known as RESONANCE
As the damping increases the resonance peak
shifts to the left because of the relation
ω[d]=ω[n]*sqrt(1-ξ^2)
$H_m(\Omega)$
Ω/ω_n
0
0.5
1
1.5
2
2.5
3
1
2
3
4
5
6



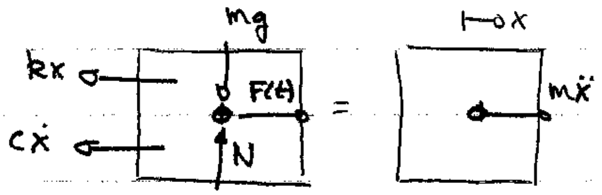
Phase Angle, ϕ
Phase Angle ϕ=tan-1[2ζ(Ω/ωn)/(1-(Ω/ωn)2)]
Ω/ωn
180
160
140
120
100
80
60
40
20
0
0
0.5
1
1.5
2
2.5
3
ζ=0
ζ=0.1
ζ=0.2
ζ=0.3
ζ=0.4
ζ=0.5
ζ=1.0

Exercice 10 (Trigonometeric notation ⇒ Forced Damped Vibration)



$F(t) = a_0 \cos(\Omega_1 t) + b_0 \sin(\Omega . t)$

Equation of motion:



$\Sigma F_x = m a_x \Rightarrow -kx - c\dot{x} + F(t) = m\ddot{x}$

$$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

The solution of the above equation is assumed to be of the form:

$$\boxed{x(t) = x_H(t) + x_p(t)}$$

where

$$\begin{cases} x_H \equiv \text{Homogeneous solution} & m\ddot{x}_H + c\dot{x}_H + kx_H = 0 \\ x_p \equiv \text{Particular solution} & m\ddot{x}_p + c\dot{x}_p + kx_p = F(t) \end{cases}$$

- The homogeneous part of the assumed solution was solved in detail in section 7 p: 32-41.

$\Delta > 0$ or $\xi > 1 \Rightarrow$ **Overdamped**, $x(t) = \bar{c}_1 e^{-(\xi + \sqrt{\xi^2 - 1})\omega_n t} + \bar{c}_2 e^{-(\xi - \sqrt{\xi^2 - 1})\omega_n t}$

$\Delta = 0$ or $\xi = 1 \Rightarrow$ **Critically Damped**, $x(t) = e^{-\omega_n t}(\bar{c}_1 + \bar{c}_2 t)$

$\Delta < 0$ or $\xi < 1 \Rightarrow$ **Underdamped**, $x(t) = e^{-\xi\omega_n t}\left[\bar{c}_1 e^{i\omega_d t} + \bar{c}_2 e^{i\omega_d t}\right]$

Assume the solution of x_p to be of the form,

$$\boxed{x_p(t) = A_p \cos(\Omega t) + B_p \sin(\Omega t)}$$

where A_p and B_p are constants to be determined.

Substituting the assumed $x_p(t)$ into $m\ddot{x}_p + c\dot{x}_p + kx_p = F(t)$

$$\ddot{x}_p = -A_p\Omega^2\cos(\Omega t) - B_p\Omega^2\sin(\Omega t)$$
$$\dot{x}_p = -A_p\Omega\sin(\Omega t) + B_p\Omega\cos(\Omega t)$$

$$m\overbrace{\left[-A_p\Omega^2\cos(\Omega t) - B_p\Omega^2\sin(\Omega t)\right]}^{\ddot{x}} \dots$$
$$+\ c\overbrace{\left[-A_p\Omega\sin(\Omega t) + B_p\Omega\cos(\Omega t)\right]}^{\dot{x}} \dots$$
$$+\ k\overbrace{\left[A_p\cos(\Omega t) + B_p\sin(\Omega t)\right]}^{x} = \overbrace{a_0\cos(\Omega t) + b_0\sin(\Omega t)}^{F(t)}$$

Rearranging,

$$\left[-mA_p\Omega^2 + c\,B_p\Omega + kA_p - a_0\right]\cos(\Omega t) \dots$$

$$+\left[-mB_p\Omega^2 - cA_p\Omega + kB_p - b_0\right]\sin(\Omega t) = 0$$

The above equation is satisfied if and only if the terms within bracket are equal to zero

(continues next page ...)

$$\begin{cases} -mA_p\Omega^2 + cB_p\Omega + kA_p - a_0 = 0 \\ -mB_p\Omega^2 - cA_p\Omega + kB_p - b_0 = 0 \end{cases}$$

Solving the above equation for A_p and B_p using $\omega_n^2 = \frac{k}{m}$

$$\boxed{\begin{aligned} A_p &= \frac{m a_0(\omega_n^2-\Omega^2) - b_0(c\Omega)}{m^2(\omega_n^2-\Omega^2)^2 + (c\Omega)^2} \\ B_p &= \frac{m b_0(\omega_n^2-\Omega^2) + a_0(c\Omega)}{m^2(\omega_n^2-\Omega^2)^2 + (c\Omega)^2} \end{aligned}}$$

Substituting A_p and B_p into $x_p = A_p\cos(\Omega t) + B_p\sin(\Omega t)$ the particular solution is obtained.

The amplitude of the particular solution is given by,

$$Amp(x_p) = |x_p| = \sqrt{A_p^2 + B_p^2} = \frac{\sqrt{a_0^2+b_0^2}}{\left[m^2(\omega_n^2-\Omega^2)^2 + (c\Omega)^2\right]^{1/2}}$$

$$\Rightarrow Amp(x_p) = |x_p| = \frac{\sqrt{a_0^2+b_0^2}}{m\omega_n^2\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}} \qquad \frac{c}{m} = 2\xi m$$

$$\Rightarrow Amp(x_p) = |x_p| = \frac{\sqrt{a_0^2+b_0^2}}{m\,H_m(\Omega)}$$

The above expression is equivalent to $|x_0| = \frac{F_0}{m} H_m(\Omega)$ on p:

(continues next page...)

The final solution of the problem is:

$$\boxed{x(t) = x_H(t) + x_p(t)}$$

where $x_H(t)$ is the homogeneous solution and $x_p(t)$ is the particular solution. The constants of integration $\bar{c}_1$ and $\bar{c}_2$ contained in the homogeneous solution $x_H(t)$ are solved by applying the initial conditions.

<u>Question</u>:

Is the homogeneous solution $x_H(t)$ of interest as $t \to \infty$

<u>Ans</u>:

For any of the three cases (critically damped, underdamped, and overdamped) the homogeneous solution $x_H(t)$ goes <u>to zero after a certain time</u>. The homogeneous soluton is not important for the forced-damped vibration problem.

Also note that in complex notation $x_0 = x_p$ and the homogeneous solution is not solved.

↳ When analyzing the forced vibrations of a system, the homogeneous (transient) solution is almost always neglected. Only the particular (steady) solution is retained.

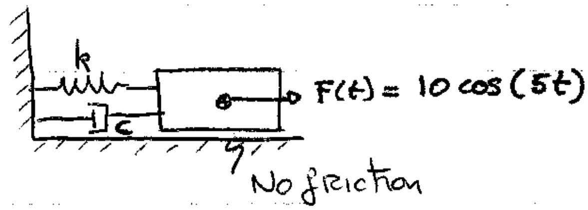
Exercice 11

Initial conditions at $t=0$ $\begin{cases} x(0)=0 \\ v(0)=0 \end{cases}$

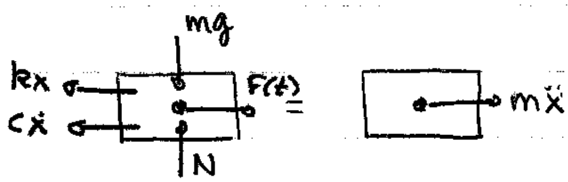
$m = 5\,kg$

$k = 10\ N/m$

$c = 1\ \frac{N.sec}{m}$



Eqt. of Motion:



$$\Sigma \bar{F} = m\bar{a} \Rightarrow -kx - c\dot{x} + F(t) = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = F(t)$$

$$\Rightarrow \ddot{x} + \frac{c}{m}\dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

$$\hookrightarrow \boxed{\ddot{x} + \frac{1}{5}\dot{x} + 2x = 2\cos(5t)}$$

where $\omega_n = \sqrt{\frac{k}{m}} = 1.414\ \frac{rad}{sec}$ is the natural frequency.

$\Omega = 5\ \frac{rad}{sec}$ is the forcing function natural frequency.

- Assume the total general solution to be of the form:

$$\boxed{x(t) = x_H(t) + x_p(t)}$$

(continues next page...)

- <u>Homogeneous solution:</u>

$$\ddot{x}_H + \frac{1}{5}\dot{x}_H + 2x_H = 0$$

Assumed $\boxed{x_H(t) = \bar{c}\, e^{-\lambda t}}$

Substituting into the homogeneous equation,

$$\hookrightarrow \quad \bar{c}\, e^{-\lambda t}\left[\lambda^2 - \frac{1}{5}\lambda + 2\right] = 0$$

The above equation is satisfied iff,

$\bar{c}\, e^{-\lambda t} = 0 \Rightarrow$ TRIVIAL SOLUTION

$\lambda^2 - \frac{1}{5}\lambda + 2 = 0$

$a^2x + bx + c = 0$

$$\hookrightarrow \quad \Delta = b^2 - 4ac = \left(-\frac{1}{5}\right)^2 - 4(1)(2) \Rightarrow \boxed{\Delta < 0} \Rightarrow \boxed{\text{Underdamped}}$$

$$\lambda_{1,2} = \frac{\frac{1}{5} \pm \sqrt{\left(-\frac{1}{5}\right)^2 - 4(1)(2)}}{2} = \frac{1}{10} \pm i\frac{\sqrt{7.96}}{2}$$

$$\Rightarrow \boxed{\lambda_{1,2} = 0.1 \pm i\, 1.4107}$$

where $Im(\lambda) = \omega_d = \sqrt{1-\xi^2}\,\omega_n$

$$\hookrightarrow \quad x_H(t) = \bar{c}\, e^{-\lambda t} = \bar{c}_1 e^{-\lambda_1 t} + \bar{c}_2 e^{-\lambda_2 t}$$

$$\Rightarrow x_H(t) = e^{-0.1t}\left[\bar{c}_1 e^{-i\omega_d t} + \bar{c}_2 e^{i\omega_d t}\right]$$

the constants $\bar{c}_1$ and $\bar{c}_2$ will be solved at the end of the problem from the initial conditions.

Particular solution:

Assume, $\boxed{x_p(t) = A_p \cos(5t) + B_p \sin(5t)}$

$$\dot{x}_p(t) = -5A_p \sin(5t) + 5B_p \cos(5t)$$
$$\ddot{x}(t) = -25A_p \cos(5t) - 25B_p \sin(5t)$$

Substituting into $m\ddot{x}_p + c\dot{x}_p + kx_p = 10\cos(5t)$

$$5\left[-25A_p \cos(5t) - 25B_p \sin(5t)\right] \ldots$$
$$+ 1\left[-5A_p \sin(5t) + 5B_p \cos(5t)\right] \ldots$$
$$+ 10\left[A_p \cos(5t) + B_p \sin(5t)\right] = 10\cos(5t)$$

Rearranging,

$$\cos(5t)\left[-125A_p + 5B_p + 10A_p - 10\right] \ldots$$
$$+ \sin(5t)\left[-125B_p - 5A_p + 10B_p - 0\right] = 0$$

The above equation is satisfied iff

$$\begin{cases} -125A_p + 5B_p + 10A_p - 10 = 0 \\ -125B_p - 5A_p + 10B_p = 0 \end{cases}$$

OR

$$\begin{cases} -115A_p + 5B_p = 10 \\ -5A_p - 115B_p = 0 \end{cases}$$

Solving for A_p and B_p,

$$\boxed{\begin{aligned} A_p &= -23/265 \simeq -0.0868 \\ B_p &= 1/265 \simeq 0.0038 \end{aligned}}$$

The final solution becomes,

$$x(t) = x_H(t) + x_p(t)$$

$$\Rightarrow x(t) = e^{-0.1t}\left[\bar{c}_1 e^{-i\omega_d t} + \bar{c}_2 e^{i\omega_d t}\right] - \frac{23}{265}\cos(5t) + \frac{1}{265}\sin(5t)$$

$$v = \frac{dx}{dt} = -0.1 e^{-0.1}\left[\bar{c}_1 e^{-i\omega_d t} + \bar{c}_2 e^{i\omega_d t}\right] + e^{-0.1t}\left[-\bar{c}_1 \omega_d i e^{-i\omega_d t} + \bar{c}_2 \omega_d i e^{i\omega_d t}\right]$$
$$\dots + \frac{23}{53}\sin(5t) + \frac{1}{53}\cos(5t).$$

Applying the initial conditions:

$$\begin{cases} x(t=0) = 0 \\ v(t=0) = 0 \end{cases}$$

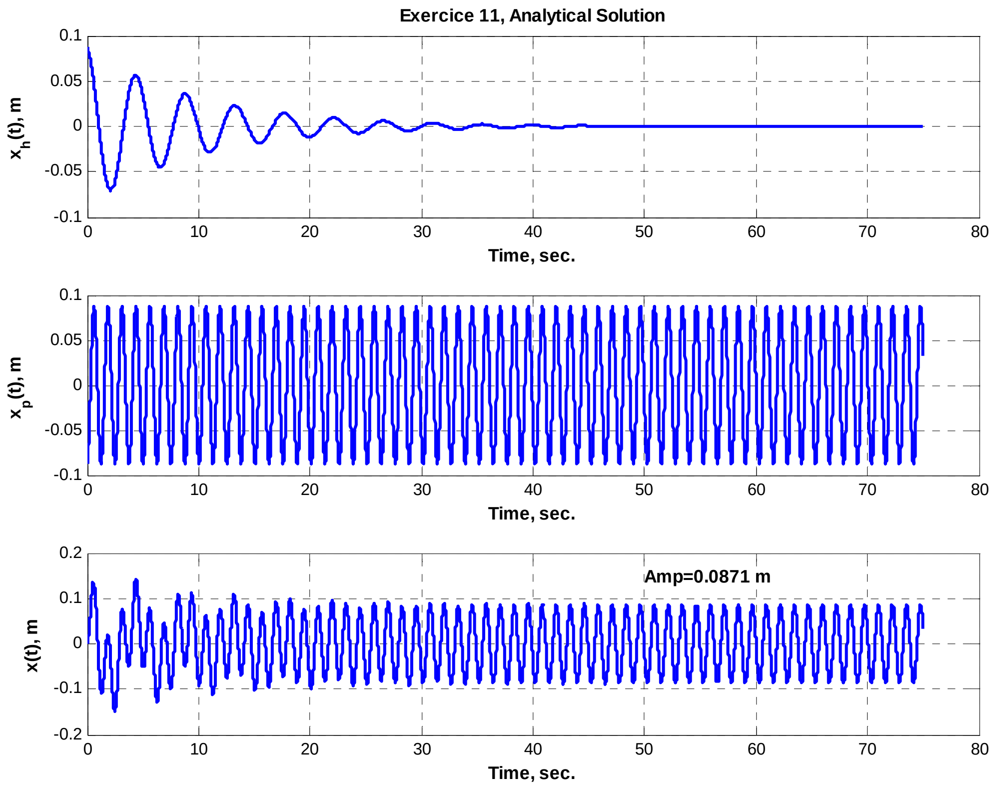
$$\hookrightarrow \begin{cases} \bar{c}_1 + \bar{c}_2 = 23/265 \\ \bar{c}_1(0.1 + i1.4107) + \bar{c}_2(0.1 - i1.4107) = \frac{1}{53} \end{cases}$$

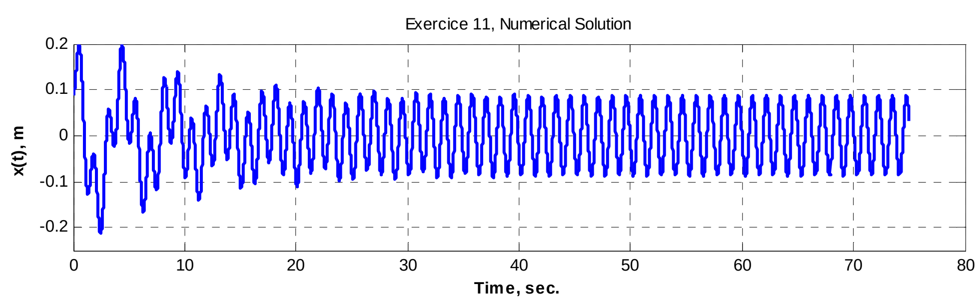
Solving for $\bar{c}_1$ and $\bar{c}_2$

$$\hookrightarrow \boxed{\begin{aligned} \bar{c}_1 &= 0.043396 - 0.0036112\,i \\ \bar{c}_2 &= 0.043396 + 0.0036112\,i \end{aligned}}$$

Substituting into $x(t)$ the complete form of the final solution is obtained.

- On the next page the analytical solution is plotted as well as the numerical.


Exercice 11, Analytical Solution
x_h(t), m
x_p(t), m
x(t), m
Amp=0.0871 m
Time, sec.


Exercice 11, Numerical Solution
x(t), m
Time, sec.

```
% dfunc1.m
function f = dfunc1(t,z)
%Input Data----------------------------------------------------------------------
----
m=5;                  % mass
k=10;                   % stiffness
cv=1;               % viscous damping

dt=0.01;                % integration time step size
n=7500;                  % number of points
tmax=dt*n;              % max. time
tspan=[0:dt:tmax];      % integration time interval from t=0~tmax

F=10*cos(5*t);
%======================================================================
===

f=zeros(2,1);
f(1)=z(2);
f(2)=F/m-cv/m*z(2)-(k/m)*z(1);

%continue
```

Exercice 12 (Solving Exercice 11 with Exponential Notation)

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

where

$$\begin{cases} m = 5\,kg \\ k = 10\,N/m \\ c = 1\,N\cdot sec/m \\ F(t) = 10\cos(5t), \quad \Omega = 5\,Rad/sec \end{cases}$$

- Assuming Harmonic Solutions

$$X(t) = X_0 e^{i\Omega t}$$
$$F(t) = F_0 e^{i\Omega t}$$

- Substituting into equation of motion

↳ $$X_0 e^{i\Omega t}\left[-m\Omega^2 + i(c\Omega) + k\right] = F_0 e^{i\Omega t}$$

⋮

see pages 57-60

⋮

$$x(t) = \frac{F_0}{m} H_m(\Omega)\, e^{i(\Omega t - \phi)}$$

Since the loading is $F = F_0\cos(\Omega t)$, the solution is given by the real part of $x(t)$

↳ $$x(t) = Re\left\{\frac{F_0}{m} H_m(\Omega)\, e^{i(\Omega t - \phi)}\right\}$$

$$\Rightarrow \boxed{x(t) = \frac{F_0}{m} H_m(\Omega)\cos(\Omega t - \phi)}$$

Numerical Application

$$x(t) = \underbrace{\frac{F_0}{m} H_m(\Omega)}_{X_0} \cos(\Omega t - \phi)$$

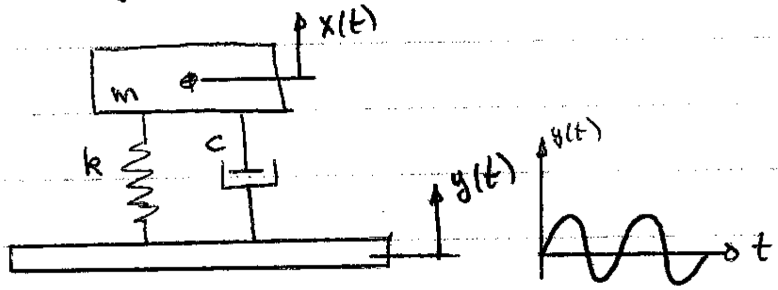
$$X_0 = \frac{F_0}{m} H_m(\Omega) = \boxed{0.0869}$$

$$\phi = -0.0435 \text{ Rad} = -2.489°$$

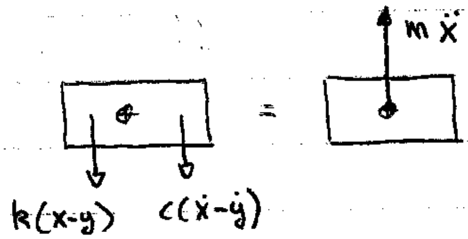
(continues next page...)

11.2 Response to Base Excitation

Consider the following system,



Eqt. of Motion:



$$\Sigma \bar{F} = m\bar{a} \Rightarrow -k(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}}$$

Solution of above differential equation of motion using exponential notation,

Assume,
$$\boxed{\begin{aligned} x(t) &= X e^{i\Omega t} \\ y(t) &= Y e^{i\Omega t} \end{aligned}}$$

Substituting into the equation of motion,

$$\hookrightarrow \quad X e^{i\Omega t}\left[-m\Omega^2 + i(c\Omega) + k\right] = Y e^{i\Omega t}\left[k + i(c\Omega)\right]$$

(continues next page...)

$$\Rightarrow \frac{X}{Y} = \frac{k + i(c\Omega)}{(k - m\Omega^2) + i(c\Omega)} \qquad \bar{z} = a + ib = |\bar{z}|e^{i\phi}$$

The magnitude $\left|\frac{X}{Y}\right|$ is equal to, $\quad \hookrightarrow |\bar{z}| = \sqrt{a^2 + b^2}$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$$\left|\frac{X}{Y}\right| = \frac{\sqrt{k^2 + (c\Omega)^2}}{\left[(k - m\Omega^2)^2 + (c\Omega)^2\right]^{1/2}} \quad \text{OR} \quad \left|\frac{X}{Y}\right| = \sqrt{\frac{k^2 + (c\Omega)^2}{(k - m\Omega^2)^2 + (c\Omega)^2}}$$

Using $\omega_n^2 = \frac{k}{m}$ and $\frac{c}{m} = 2\xi\omega_n$

$$\hookrightarrow \left|\frac{X}{Y}\right| = \frac{\cancel{m}\omega_n^2 \sqrt{1 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}}{\cancel{m}\omega_n^2\left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

$$\Rightarrow \boxed{\left|\frac{X}{Y}\right| = \sqrt{1 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}\; H_m(\Omega)\, \omega_n^2}$$

where $H_m(\Omega) = \dfrac{1}{\omega_n^2\left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$

- <u>Phase Angle</u>:

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}(X/Y)}{\mathrm{Re}(X/Y)}\right)$$

$$\hookrightarrow \frac{X}{Y} = \frac{[k + i(c\Omega)][(k - m\Omega^2) - i(c\Omega)]}{[(k - m\Omega^2) + i(c\Omega)][(k - m\Omega^2) - i(c\Omega)]}$$

$$\Rightarrow \frac{X}{Y} = \frac{k(k - m\Omega^2) - i(kc\Omega) + i(c\Omega)(k - m\Omega^2) + (c\Omega)^2}{(k - m\Omega^2)^2 + (c\Omega)^2}$$

(continues next page...)

$$\Rightarrow \frac{X}{Y} = \frac{1}{(k-m\Omega^2)^2+(c\Omega)^2}\left\{\left[k(k-m\Omega^2)+(c\Omega)^2\right]+i\left[-\cancel{kc\Omega}+\cancel{kc\Omega}-mc\Omega^3\right]\right\}$$

$$\Rightarrow \frac{X}{Y} = \frac{1}{(k-m\Omega^2)^2+(c\Omega)^2}\left\{\underbrace{\left[k(k-m\Omega^2)+(c\Omega)^2\right]}_{Re}-\underbrace{i(mc\Omega^3)}_{Im}\right\}$$

$$\hookrightarrow \boxed{\phi = \tan^{-1}\left[\frac{-mc\Omega^3}{k(k-m\Omega^2)+(c\Omega)^2}\right]}$$

OR using $\omega_n^2 = k/m$ and $\frac{c}{m} = 2\xi\omega_n$,

$$\hookrightarrow \boxed{\phi = \tan^{-1}\left[\frac{-2\xi\left(\frac{\Omega}{\omega_n}\right)^3}{1+(4\xi^2-1)\left(\frac{\Omega}{\omega_n}\right)^2}\right]}$$

The ratio of the amplitude of the response $x(t)$ to that of the base excitation $y(t)$, $|X/Y|$, is called the displacement transmissibility. The variations of $|X/Y|$ and ϕ are given by

$$\begin{cases} \dfrac{X}{Y} = \sqrt{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}\; H_m(\Omega)\,\omega_n^2 \\[2ex] \phi = \tan^{-1}\left[\dfrac{-2\xi\left(\frac{\Omega}{\omega_n}\right)^3}{1+(4\xi^2-1)\left(\frac{\Omega}{\omega_n}\right)^2}\right] \end{cases}$$

and are shown on the figures next page.

ME-400

Algebra

$\omega_n^2 = k/m \quad , \quad \frac{c}{m} = 2\xi\omega_n$

$$\frac{m c \Omega^3}{k(k - m\Omega^2) + (c\Omega)^2}$$

$$= \frac{m^2\, 2\xi\omega_n \Omega^3}{k^2\left(1 - \frac{\Omega^2}{\omega_n^2}\right) + 4\xi^2 m^2 \omega_n^2 \Omega^2}$$

$$= \frac{\cancel{m^2} \cdot 2\xi\omega_n\Omega^3}{\cancel{m^2}\left[\left(\frac{k}{m}\right)^2\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right) + 4\xi^2\omega_n^2\Omega^2\right]}$$

$$\left(\frac{k}{m}\right)^2 = \left(\omega_n^2\right)^2 = \omega_n^4$$

$$= \frac{2\xi \cancel{\omega_n} \Omega^3}{\omega_n^4\left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right) + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right.}$$

$$= \frac{2\xi\left(\frac{\Omega}{\omega_n}\right)^3}{\left(\frac{\Omega}{\omega_n}\right)^2\left(4\xi^2 - 1\right) + 1}$$

The form of the displacement response becomes,

$$x(t) = \bar{X} e^{i\Omega t}$$

where using the complex notation $\bar{z} = a - ib = |\bar{z}| e^{-i\phi}$, and recalling that

$$\left|\frac{\bar{X}}{\bar{Y}}\right| = \sqrt{1 + 4\xi^2 \left(\frac{\Omega}{\omega_n}\right)^2}\, H_m(\Omega)\, \omega_n^2$$

$$\Rightarrow \quad \bar{X} = \bar{Y} \sqrt{1 + 4\xi^2 \left(\frac{\Omega}{\omega_n}\right)^2}\, H_m(\Omega)\, \omega_n^2\, e^{-i\phi}$$

so that,

$$x(t) = \bar{X} e^{i\Omega t}$$

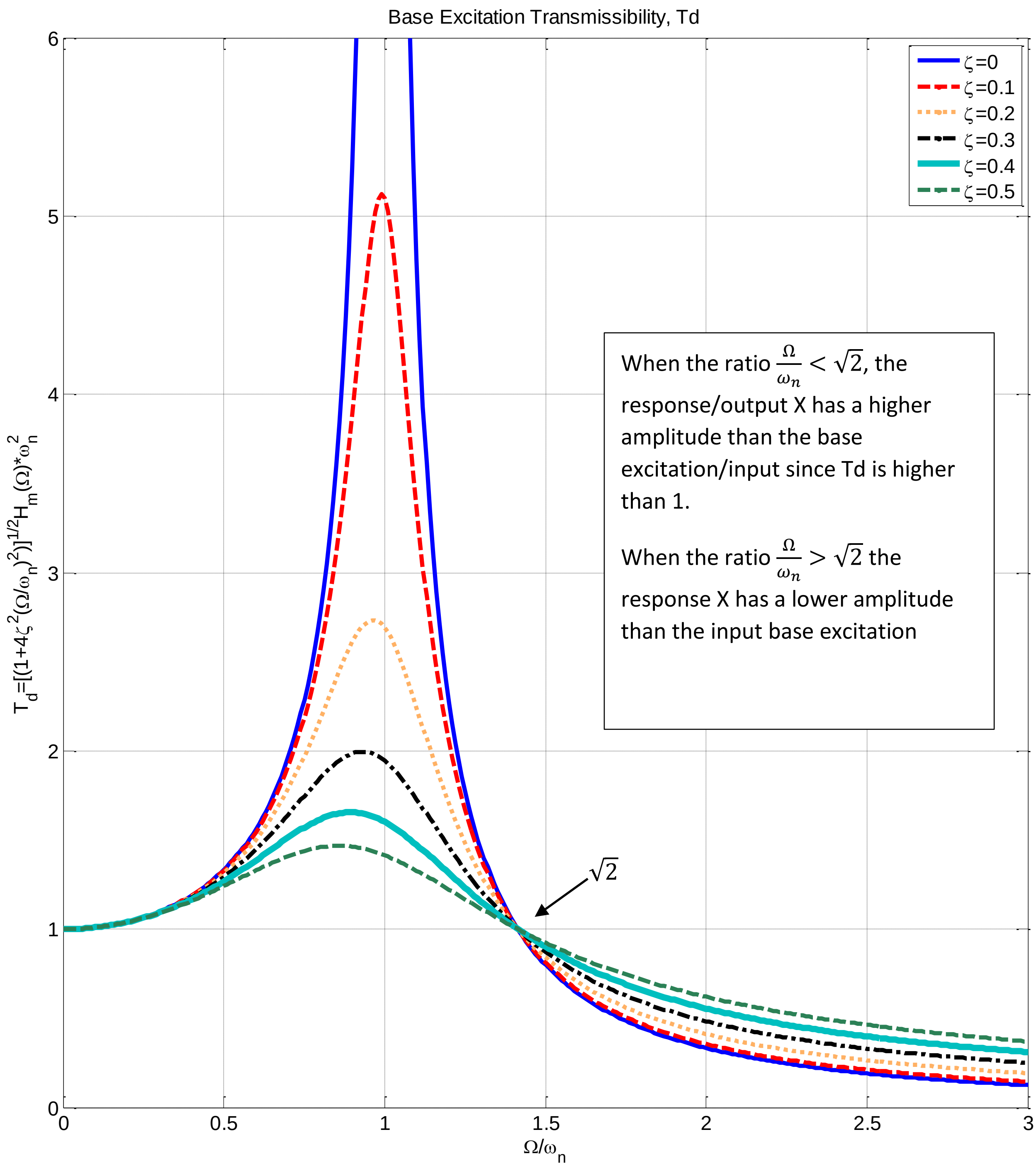
$$\Rightarrow \quad x(t) = \bar{Y} \sqrt{1 + 4\xi^2 \left(\frac{\Omega}{\omega_n}\right)^2}\, H_m(\Omega)\, \omega_n^2\, e^{-i\phi} e^{i\Omega t}$$

Using $e^{a+b} = e^a \cdot e^b$

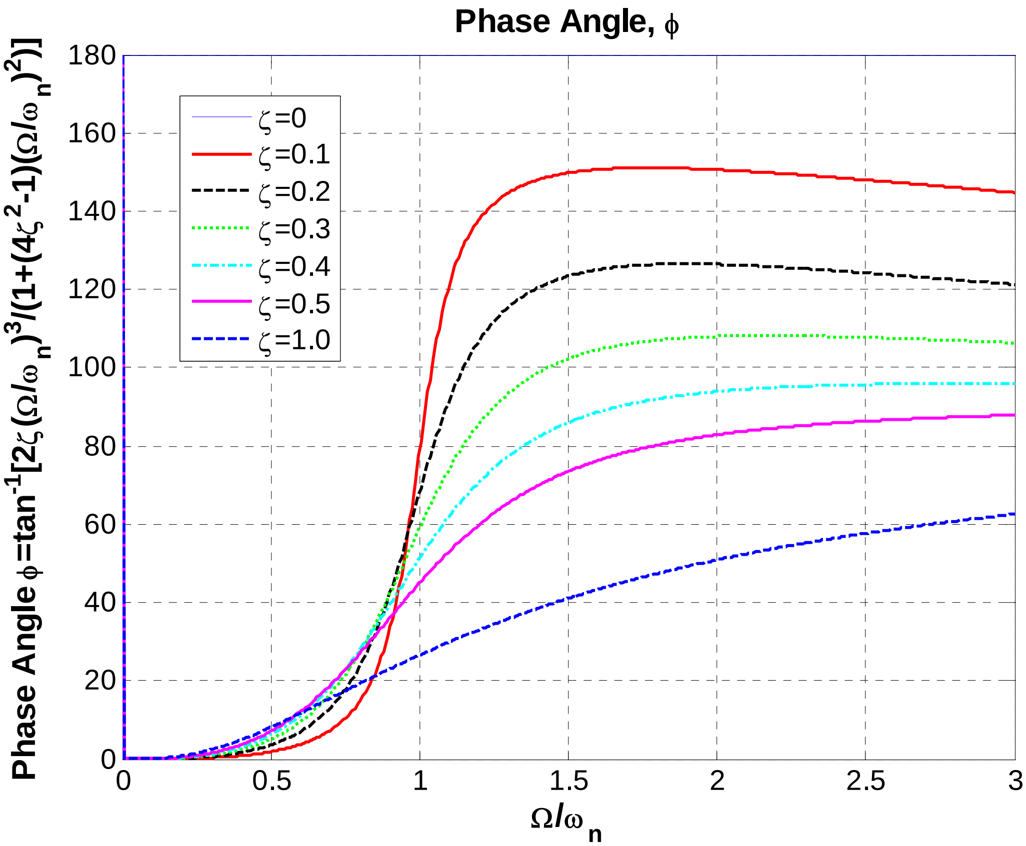
$$\boxed{x(t) = \bar{Y} \sqrt{1 + 4\xi^2 \left(\frac{\Omega}{\omega_n}\right)^2}\, H_m(\Omega)\, \omega_n^2\, e^{i(\Omega t - \phi)}}$$

where

$$\begin{cases} \phi = \tan^{-1}\left[\dfrac{-2\xi\left(\frac{\Omega}{\omega_n}\right)^3}{1 + (4\xi^2 - 1)\left(\frac{\Omega}{\omega_n}\right)^2}\right] \\[2ex] H_m(\Omega) = \dfrac{1}{\omega_n^2\left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}} \end{cases}$$



Base Excitation Transmissibility, Td
$T_d=[(1+4\zeta^2(\Omega/\omega_n)^2)]^{1/2}H_m(\Omega)*\omega_n^2$
Ω/ω_n
$\zeta=0$
$\zeta=0.1$
$\zeta=0.2$
$\zeta=0.3$
$\zeta=0.4$
$\zeta=0.5$
$\sqrt{2}$
When the ratio $\frac{\Omega}{\omega_n} < \sqrt{2}$, the response/output X has a higher amplitude than the base excitation/input since Td is higher than 1.
When the ratio $\frac{\Omega}{\omega_n} > \sqrt{2}$ the response X has a lower amplitude than the input base excitation


Phase Angle, φ
Phase Angle φ=tan⁻¹[2ζ(Ω/ωn)³/(1+(4ζ²-1)(Ω/ωn)²)]
Ω/ωn
ζ=0
ζ=0.1
ζ=0.2
ζ=0.3
ζ=0.4
ζ=0.5
ζ=1.0
0
20
40
60
80
100
120
140
160
180
0
0.5
1
1.5
2
2.5
3

The response of the block 'm' is given by,

$$\boxed{\begin{aligned} x(t) &= X e^{i\Omega t} \\ &= \sqrt{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2} \cdot Y \, H_m(\Omega)\, \omega_n^2 \, e^{i\Omega t} \end{aligned}}$$

and the response of the base by,

$$\boxed{\begin{aligned} y(t) &= Y e^{i\Omega t} \\ &= \frac{X e^{i\Omega t}}{\sqrt{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2} \cdot H_m(\Omega) \cdot \omega_n^2} \end{aligned}}$$

11.3 Force Transmitted to Base

In the case of the base excitation a force F is transmitted to the base from the spring and the viscous damper. That force is equal to,

$$\boxed{F = -m\ddot{x} = k(x-y) + c(\dot{x}-\dot{y})}$$

$\rightarrow$ $F(t) = -m\ddot{x}(t) = +m\Omega^2 X e^{i\Omega t}$

$\Rightarrow$ $\dfrac{F(t)}{k} = \dfrac{m}{k}\Omega^2 X e^{i\Omega t}$, using $\omega_n^2 = k/m$

$\Rightarrow$ $\dfrac{F(t)}{X k} = \left(\dfrac{\Omega}{\omega_n}\right)^2 e^{i\Omega t}$, using $\dfrac{X}{Y} = \sqrt{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}\, H_m(\Omega)\, \omega_n^2$

OR

$$\frac{F(t)}{Y k} = \left(\frac{\Omega}{\omega_n}\right)^2 \sqrt{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2} \cdot H_m(\Omega)\,\omega_n^2 \, e^{i\Omega t}$$

If F_T is the amplitude or maximum value of $F(t)$,

$$\hookrightarrow \left|\frac{F(t)}{Yk}\right| = \frac{F_T}{Yk} = \left|\left(\frac{\Omega}{\omega_n}\right)^2 \cdot \sqrt{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}\ \overset{\omega_n^2}{H_m(\Omega)} Y e^{i\Omega t}\right| \quad ; \quad |e^{i\Omega t}| = 1$$

$$\Rightarrow \boxed{\frac{F_T}{Yk} = \left(\frac{\Omega}{\omega_n}\right)^2 \left[\frac{1+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2}\right]^{1/2}}$$

The plot of the above expression is shown on the next page.

Note that the transmitted force is in phase with the motion of the block mass m.

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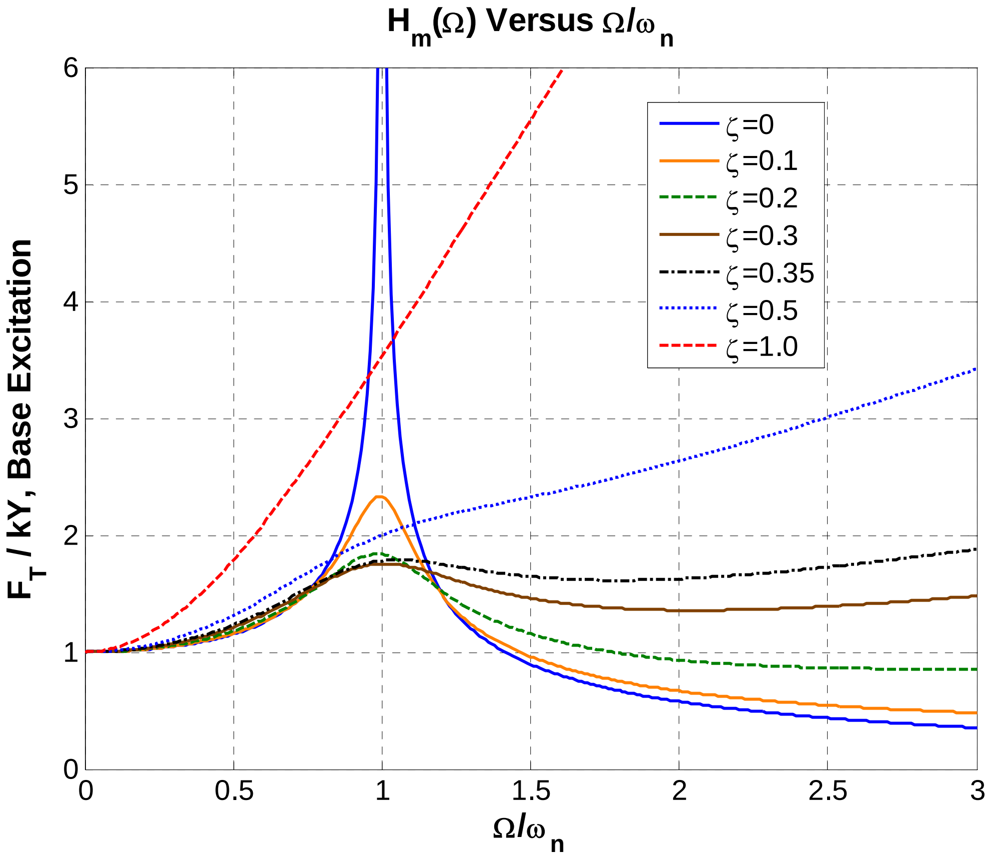
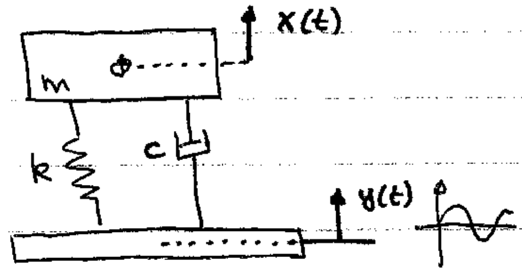

H_m(Ω) Versus Ω/ω_n
F_T / kY, Base Excitation
Ω/ω_n
ζ=0
ζ=0.1
ζ=0.2
ζ=0.3
ζ=0.35
ζ=0.5
ζ=1.0
0
0.5
1
1.5
2
2.5
3
4
5
6

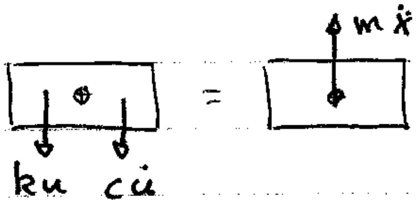
Supplement for Base Excitation

Using the notation for relative displacement,

$$\boxed{u(t) = x(t) - y(t)}$$



Eqt. of Motion:



$$\Sigma F = ma \Rightarrow \quad -ku - c\dot{u} = m\ddot{x} = m(\ddot{u} + \ddot{y})$$

$$\Rightarrow \quad \boxed{m\ddot{u} + c\dot{u} + ku = m\ddot{y}}$$

Assume $\quad u(t) = U e^{i\Omega t}$

$\qquad\qquad\; y(t) = Y e^{i\Omega t}$

Substituting,

$$U \cancel{e^{i\Omega t}}\left[-m\Omega^2 + i(c\Omega) + k\right] = Y \cancel{e^{i\Omega t}}\left[-m\Omega^2\right]$$

$$\Rightarrow \quad \frac{U}{Y} = \frac{-m\Omega^2}{\left[(k - m\Omega^2) + i(c\Omega)\right]}$$

The magnitude $\left|\frac{U}{Y}\right|$ is equal to:

$$\left|\frac{U}{Y}\right| = \frac{\sqrt{(m\Omega^2)^2}}{\left[(k-m\Omega^2)^2+(c\Omega)^2\right]^{1/2}}$$

Using $\omega_n^2 = \frac{k}{m}$ and $\frac{c}{m} = 2\xi\omega_n$

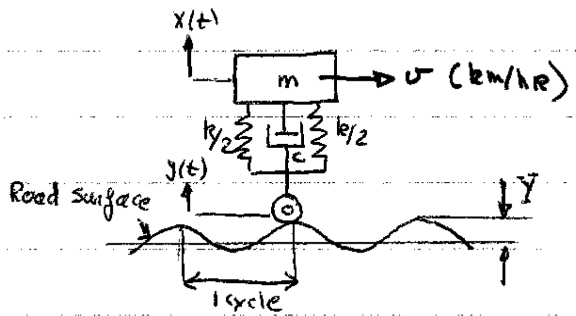
$$\left|\frac{U}{Y}\right| = \frac{m\Omega^2}{k\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

$$\Rightarrow \boxed{\left|\frac{U}{Y}\right| = \left(\frac{\Omega}{\omega_n}\right)^2 \cdot \frac{1}{\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2+4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}}$$

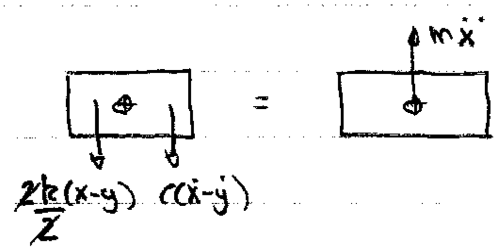
Exercice 13 (Automobile suspension)

Consider an automobile traveling over a rough road at a speed v (km/hr). The suspension system has a spring constant $k = 40$ kN/m and a damping $\xi = 0.1$. The road surface varies sinusoidally with an amplitude $Y = 0.05$ m and a wavelength $L = 6$ m.
Write a Matlab code to find the displacement amplitude of the automobile for the following conditions

(a) mass of automobile = 600 kg (empty), 1000 kg (loaded)
(b) velocity of automobile, 10 km/hr, 50 km/hr, and 100 km/hr



Free-Body Diagram:



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<u>Equation of motion</u>

$$\Sigma \bar{F} = m\bar{a} \Rightarrow -k(x-y) - c(\dot{x}-\dot{y}) = m\ddot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}}$$

Assuming $\begin{cases} x(t) = X e^{i\Omega t} \\ y(t) = Y e^{i\Omega t} \end{cases}$

and substituting into the equation of motion

$$\hookrightarrow X \cancel{e^{i\Omega t}} \left[-m\Omega^2 + i(c\Omega) + k \right] = Y \cancel{e^{i\Omega t}} \left[k + i(c\Omega) \right]$$

$$\Rightarrow \frac{X}{Y} = \frac{k + i(c\Omega)}{(k - m\Omega)^2 + i(c\Omega)}$$

OR

$$\left| \frac{X}{Y} \right| = \sqrt{1 + 4\xi^2 \left(\frac{\Omega}{\omega_n}\right)^2} \; H_m(\Omega) \cdot \omega_n^2$$

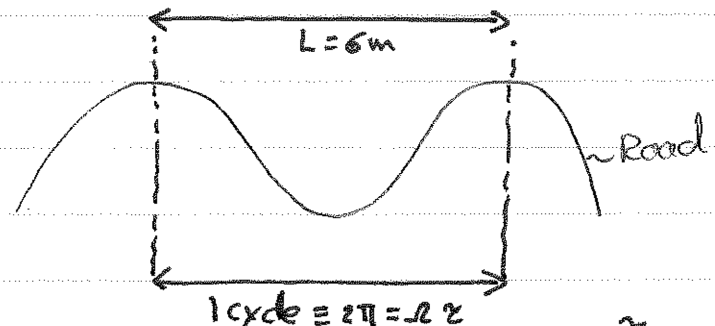
where $H_m(\Omega) = \dfrac{1}{\omega_n^2 \left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2 \left(\frac{\Omega}{\omega_n}\right)^2 \right]^{1/2}}$

where $\omega_n^2 = \dfrac{k}{m} = \dfrac{40 \cdot 10^3}{600}$ sample calculation for $m = 600$ kg and $v = 10 \frac{km}{hr} = 2.778 \frac{m}{sec}$.

$$\Rightarrow \boxed{\omega_n = 8.1649 \text{ rad/sec}}$$

(continues next page...)

- How to determine Ω :



τ = period (sec)

frequency , $f \equiv \frac{1}{sec} = \frac{v}{L} \equiv \frac{m/sec}{m} = \frac{1}{sec}$

$\hookrightarrow \quad f = \frac{2.778}{6} = 0.463$

$\hookrightarrow \quad \Omega = 2\pi f \Rightarrow \boxed{\Omega = 2.909 \ \frac{Rad}{sec}}$

Substituting into,

$$\left|\frac{X}{Y}\right| = \sqrt{1+4\zeta^2\left(\frac{\Omega}{\omega_n}\right)^2} \ H_m(\Omega)\,\omega_n^2$$

$$\Rightarrow \boxed{\left|\frac{X}{Y}\right| = 1.1445 \ \ m}$$

Using the attached MATLAB code, the results for the other cases are tabulated below

$m = 600$ kg
- $X = 0.0572$ for $\omega_n = 8.1650$ and $\Omega = 2.9089$ (10 km/hr)
- $X = 0.0241$ for $\omega_n = 8.1650$ and $\Omega = 14.5444$ (50 km/hr)
- $X = 0.0052$ for $\omega_n = 8.1650$ and $\Omega = 29.0888$ (100 km/hr)

$m = 1000$ kg
- $X = 0.0633$ for $\omega_n = 6.3246$ and $\Omega = 2.9089$
- $X = 0.0128$ for $\omega_n = 6.3246$ and $\Omega = 14.5444$
- $X = 0.0034$ for $\omega_n = 6.3246$ and $\Omega = 29.0888$

From the table it is obsewed that at the lowest velocity of 10 km/hr the amplitude $|X|$ is larger than the amplitude of the "bump" on the road $|Y| = 0.05$ m.

As the speed increases, the frequency Ω also increases and the amplitude $|X|$ reduces significantly.

(see attached figure)

```
% Exercice 13
clear all
clc

k=40e3;
zeta=0.1;
L=6;
Y=0.05

m=[600;1000];
v=[10;50;100]*1000/3600;

for i=1:2
    M=m(i);
    omega_n(i)=sqrt(k/M);
    for j=1:3
        V=v(j);
        Omega(j)=2*pi*V/L;
        r=Omega(j)/omega_n(i);
        H_m=1/((1-r^2)^2+4*zeta^2*r^2)^0.5;
        X(j)=Y*sqrt(1+4*zeta^2*r^2)*H_m;
    end

    X_amp(:,i)=X;
end


X_amp

V=v*3600/1000;
plot(V,X_amp(:,1),'-rs',V,X_amp(:,2),':bs')
ylabel('Amplitude, |X| (m)')
xlabel('Velocity V (km/hr)')
legend('\omega_n=8.1650 rad/sec','\omega_n=6.3246 rad/sec')
grid on

data=[Omega' X_amp]
```

Calculations Check for m=600 kg, v=10 km/hr

> $restart;$

> $with(MTM):$

> $L := 6; \text{pi} := 3.14156;; V := 10; Y := 0.05; \text{xi} := 0.1; m := 600; k := 40\text{e}3; \text{Omega} := \frac{2 \cdot \text{pi} \cdot \left(\frac{V \cdot 1000}{3600}\right) \cdot 1}{L}; \text{omega}[n] := \text{sqrt}\left(\frac{k}{m}\right); c := 2 \cdot \text{xi} \cdot m \cdot \text{omega}[n]; r := \frac{\text{Omega}}{\text{omega}[n]};$

$$L := 6$$
$$\pi := 3.14156$$
$$V := 10$$
$$Y := 0.05$$
$$\xi := 0.1$$
$$m := 600$$
$$k := 40000.$$
$$\Omega := 2.908851852$$
$$\omega_n := 8.164965809$$
$$c := 979.7958970$$
$$r := 0.3562601387$$

> $X\_Y := Complex\left(\frac{\left(k \cdot \left(k - m \cdot \Omega^2\right) + (c \cdot \text{Omega})^2\right)}{\left(k - m \cdot \Omega^2\right)^2 + (c \cdot \text{Omega})^2}, \frac{-\left(m \cdot c \cdot \Omega^3\right)}{\left(k - m \cdot \Omega^2\right)^2 + (c \cdot \text{Omega})^2}\right);$

$$X\_Y := 1.144410299 - 0.01178533666\,\mathrm{I}$$

> $RE := \text{Re}(X\_Y);$

$$RE := 1.144410299$$

> $IM := \text{Im}(X\_Y);$

$$IM := -0.01178533666$$

> $Amp := \text{sqrt}\left(RE^2 + IM^2\right); \text{Phi} := atan\left(\frac{IM}{RE}\right);$

$$Amp := 1.144470981$$
$$\Phi := -0.01029781021$$

> $convert(X\_Y, \text{polar});$

$$\text{polar}(1.144470981, -0.01029781021)$$

> $X := Y \cdot Amp;$

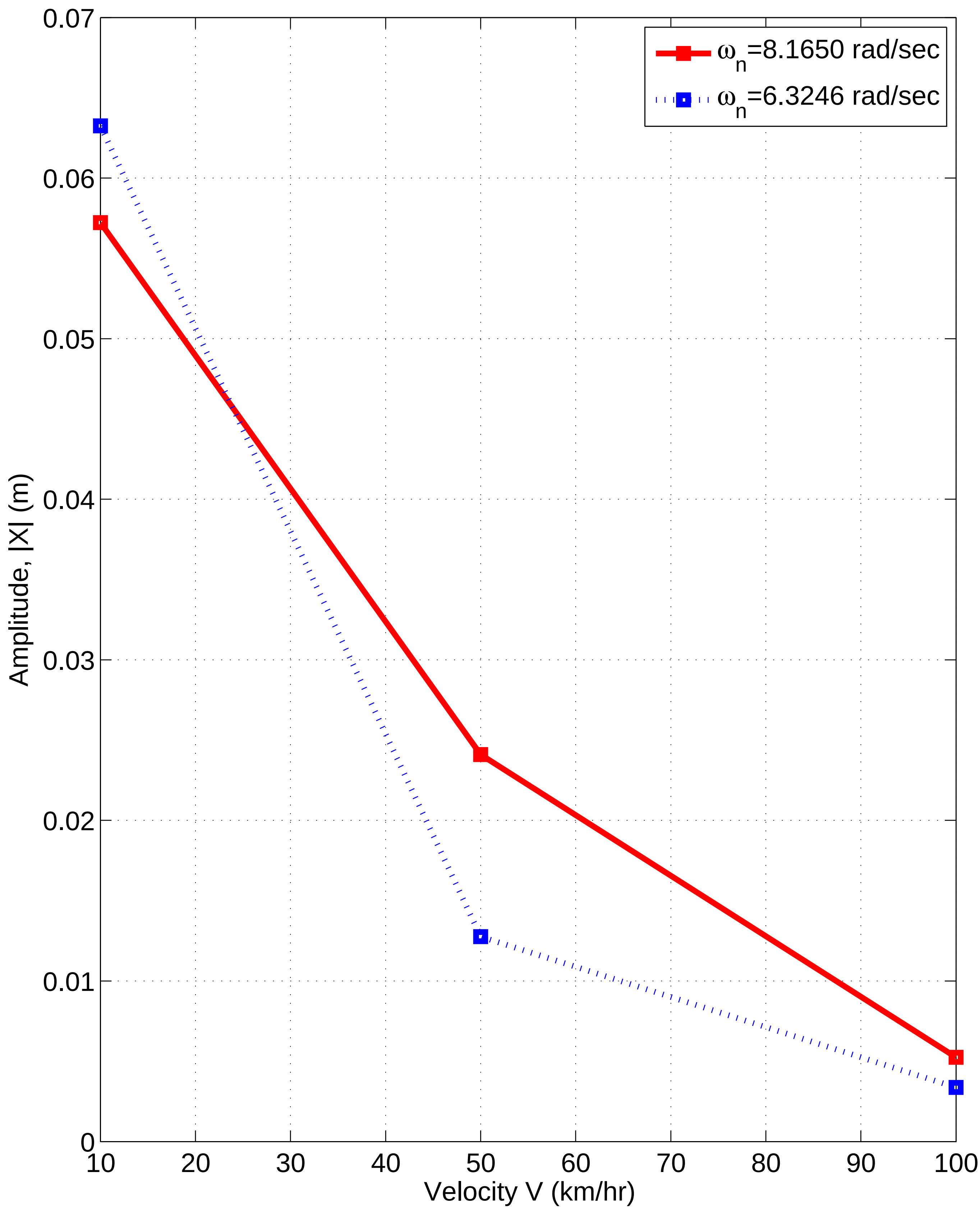
$$X := 0.05722354905$$

> $H\_m := 1/\left((1-r\text{^}2)\text{^}2 + 4*\xi^2*r\text{^}2\right)\text{^}0.5;$

$$H\_m := 1.141576841$$

> $X := Y*\text{sqrt}(1 + 4*\text{xi}\text{^}2*r\text{^}2)*H\_m;$

$$X := 0.05722354902$$

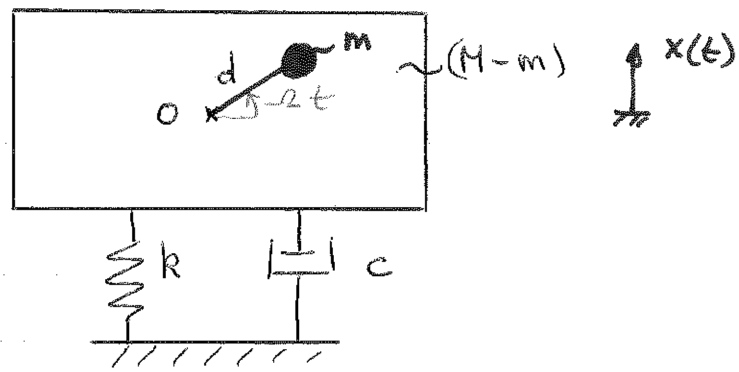


ωn=8.1650 rad/sec
ωn=6.3246 rad/sec
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Amplitude, |X| (m)
10
20
30
40
50
60
70
80
90
100
Velocity V (km/hr)

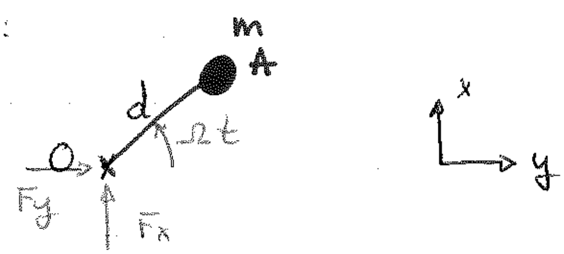
11.3 Response under Rotating Unbalance

Unbalance in machinery is one of the main causes of vibration. A simplified model is presented in this section.

$m \equiv$ eccentric mass



For the eccentric mass:



$$\sum \bar{F} = m\,\bar{a}_A$$

$$\Rightarrow \begin{pmatrix} F_x \\ F_y \end{pmatrix} = m\left[\bar{a}_O + \bar{\alpha} \times \bar{R}_{A/O} - \cancel{\omega^2}\,\Omega^2\,\bar{R}_{A/O}\right]$$

Assuming constant angular velocity Ω

$$\alpha = \frac{d\Omega}{dt} = 0$$

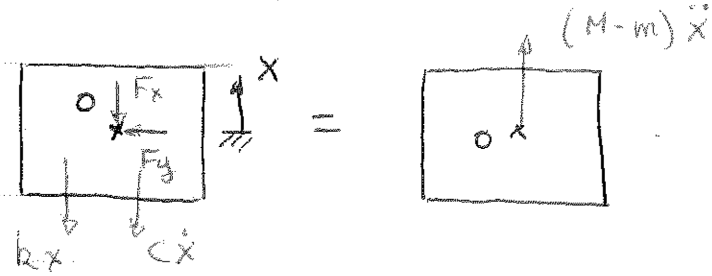
$$= m\left[\begin{pmatrix} \ddot{x} \\ 0 \end{pmatrix} - \Omega^2 \begin{pmatrix} d\sin(\Omega t) \\ d\cos(\Omega t) \end{pmatrix}\right]$$

$$\hookrightarrow \begin{cases} F_x = m\ddot{x} - m\Omega^2 d\sin(\Omega t) & \text{— (i)} \\ F_y = -m\Omega^2 d\cos(\Omega t) & \text{— (ii)} \end{cases}$$

(continues next page...)

Considering the whole system,

it is assumed that system is at equilibrium.



$\Sigma \bar{F} = m\bar{a}$

(x-dir) $\Rightarrow \; -kx - c\dot{x} - F_x = (M-m)\ddot{x}$

$$\Rightarrow \; -kx - c\dot{x} - \cancel{m\ddot{x}} - m\Omega^2 d \sin(\Omega t) = M\ddot{x} - \cancel{m\ddot{x}}$$

$$\Rightarrow \; \boxed{M\ddot{x} + c\dot{x} + kx = m\Omega^2 d \sin(\Omega t)}$$

Equation of Motion for Response under Rotating Unbalance

Using complex notation:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\hookrightarrow \quad \boxed{M\ddot{x} + c\dot{x} + kx = m\Omega^2 d \, \mathrm{Im}[e^{i\Omega t}]}$$

Neglecting homogeneous solution and assuming harmonic solution for above equation

$$\boxed{x(t) = X e^{i\Omega t} = X \, \mathrm{Im}[e^{i\Omega t}]}$$

the equation of motion can be rewritten as

$$X \, \mathrm{Im}[\cancel{e^{i\Omega t}}][-\Omega^2 M + i(c\Omega) + k] = m\Omega^2 d \, \mathrm{Im}[\cancel{e^{i\Omega t}}]$$

$$\Rightarrow \quad X = \frac{m\Omega^2 d}{(k - M\Omega^2) + i(c\Omega)}$$

Following same procedure as in p: 58,

$$\bar{X} = \frac{md\Omega^2}{(k-M\Omega^2)+i(c\Omega)} = \frac{md\Omega^2\left[(k-M\Omega^2)-i(c\Omega)\right]}{\left[(k-M\Omega^2)^2+(c\Omega)^2\right]}$$

$$\Rightarrow \bar{X} = \frac{md\Omega^2}{\left[(k-M\Omega^2)^2+(c\Omega)^2\right]}\left[(k-M\Omega^2)-i(c\Omega)\right]$$

Using $\bar{z} = a+ib = |\bar{z}|\,e^{i\phi}$, where $|\bar{z}| = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

$$\bar{X} = \frac{md\Omega^2}{\left[(k-M\Omega^2)^2+(c\Omega)^2\right]}\cdot\left[(k-M\Omega^2)^2+(c\Omega)^2\right]^{1/2} e^{-i\phi}$$

$$\Rightarrow \bar{X} = \frac{md\Omega^2}{\left[(k-M\Omega^2)^2+(c\Omega)^2\right]^{1/2}}\,e^{-i\phi}$$

Substituting into assumed solution $\boxed{x = \bar{X}\,e^{i\Omega t}}$

$$x(t) = \frac{md\Omega^2}{\left[(k-M\Omega^2)^2+(c\Omega)^2\right]^{1/2}}\,e^{i\Omega t}\cdot e^{-i\phi}$$

$$\Rightarrow \boxed{x(t) = \frac{md\Omega^2}{\left[(k-M\Omega^2)^2+(c\Omega)^2\right]^{1/2}}\,e^{i(\Omega t-\phi)}}$$

where $\phi = \tan^{-1}\left[\frac{-(c\Omega)^2}{(k-M\Omega^2)^2}\right]$

Using the notation

$$\omega_n^2 = \frac{k}{M} \quad \& \quad \frac{c}{M} = 2\xi\omega_n$$

$$(k - M\Omega^2)^2 + (c\Omega)^2 = M^2\left\{\left(\frac{k}{M} - \Omega^2\right)^2 + \left(\frac{c}{M}\Omega\right)^2\right\}$$

$$= M^2\left\{(\omega_n^2 - \Omega^2)^2 + (2\xi\omega_n\Omega)^2\right\}$$

$$= M^2\omega_n^4\left\{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right\}$$

the expression for $x(t)$ becomes,

$$x(t) = \frac{md\Omega^2}{M\omega_n^2\left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right) + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}} e^{i(\Omega t - \phi)}$$

OR

$$\boxed{x(t) = \frac{md}{M}\Omega^2 H_m(\Omega)\, e^{i(\Omega t - \phi)}}$$

where

$$H_m(\Omega) = \frac{1}{\omega_n^2\left[\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

and $$\tan\phi = \frac{-c\Omega}{k - M\Omega^2} = \frac{-c\Omega}{M\left(\frac{k}{M} - \Omega^2\right)} = \frac{-\frac{c}{M}\Omega}{M(\omega_n^2 - \Omega^2)}$$

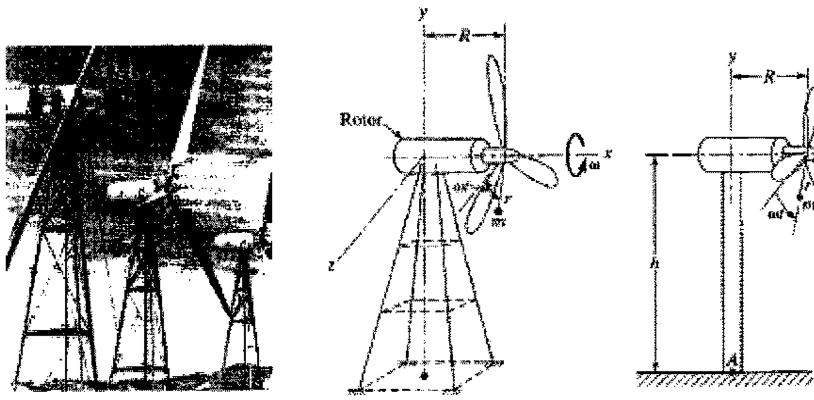
$$\Rightarrow \tan\phi = \frac{-2\xi\omega_n\Omega}{M\omega_n^2\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)} = \frac{-2\xi\left(\frac{\Omega}{\omega_n}\right)}{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right)}$$

OR $$\phi = \tan^{-1}\left(\frac{-2\xi\left(\frac{\Omega}{\omega_n}\right)}{1 - \left(\frac{\Omega}{\omega_n}\right)^2}\right) = -\tan^{-1}\left(\frac{2\xi\left(\frac{\Omega}{\omega_n}\right)}{1 - \left(\frac{\Omega}{\omega_n}\right)^2}\right)$$

Example: (Wind Turbine)

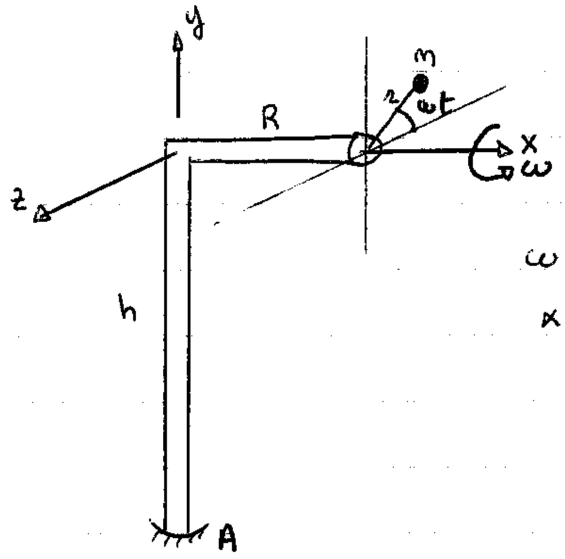
A three-bladed wind turbine has small unbalanced mass *m* located at a radius *r* in the plane of the blades. The blades are located from the central vertical y-axis at a distance *R* and rotate at an angular velocity of ω. If the supporting truss can be modeled as a hollow steel shaft of outer diameter 0.1 m and inner diameter 0.08 m, determine the maximum stresses developed at the base of the support (point A). The mass moment of inertia of the turbine system about the vertical y-axis is J_0.

Assume R=0.5 m, m=0.1 kg, r=0.1 m, J_0=100 kg-m^2, h=8 m, and $\omega = 31.416$ rad/sec.



(a) (b)

Exercice



$\omega = const$

$\alpha = \frac{d\omega}{dt} = 0$

- Force created by unbalance mass m,

$$\bar{F} = \begin{pmatrix} 0 \\ m r \omega^2 \sin(\omega t) \\ m r \omega^2 \cos(\omega t) \end{pmatrix}$$

$\hookrightarrow$ magnitude $= |\bar{F}| = \sqrt{F_y^2 + F_z^2} \Rightarrow \boxed{|\bar{F}| = m r \omega^2}$

- <u>Bending stress at point A</u>

$$\sigma_b = \frac{M_z . c}{I_z} \quad \text{where} \quad \begin{cases} M_z = |\bar{F}| . R = m r \omega^2 R \\ c = d_o/2 \text{ (max. bending stress at surface of beam)} \\ I_z = \frac{\pi}{64} (d_o^4 - d_i^4) \end{cases}$$

$$\hookrightarrow \sigma_b = \frac{m r \omega^2 . R \, (d_o/2)}{\frac{\pi}{64} (d_o^4 - d_i^4)} \Rightarrow \boxed{\sigma_b = 85\,124 \text{ kPa}}$$

(continues next page ...)

- Stress due to torsion at point A

$$\sigma_t = \frac{T \cdot R}{J} = \frac{M_y (d_o/2)}{\frac{\pi}{32}(d_o^4 - d_i^4)}$$

$$\Rightarrow \sigma_t = \frac{m r \omega^2 R \,(d_o/2)}{\frac{\pi}{32}(d_o^4 - d_i^4)} = \frac{\sigma_b}{2}$$

$$\Rightarrow \boxed{\sigma_t = 42.564 \text{ kPa}}$$

This example is a good illustration for structural-dynamics, in other words finding the stresses induced by the motion/vibrations of a structural system.

12. Stability Analysis and Self-Excitation

The force acting on a vibrating system is usually external to the system and independent of the motion. However, there are systems for which the exciting force is a function of the motion, such as displacement, velocity, or acceleration. Such systems are called **self-excited** systems since the motion itself produces the exciting force.

The instability of rotating shafts, flutter of wings/airfoils, flow induced vibration of pipes, and the automobile wheel shimmy are typical examples of self-excited systems.

12.1 Dynamic stability Analysis

Considering the following equation of motion,

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Assuming the solution : $\boxed{x(t) = \bar{c}e^{-\lambda t}}$

Substituting into the equation of motion

$$\rightarrow \bar{c}e^{-\lambda t}\left[\lambda^2 m - \lambda c + k\right] = 0$$

The above equation is satisfied iff

$\bar{c}e^{-\lambda t} = 0 \Rightarrow$ TRIVIAL SOLUTION

$m\lambda^2 - c\lambda + k = 0 \Rightarrow$ CHARACTERISTIC EQUATION

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{c \pm \sqrt{c^2 - 4km}}{2m}$$

OR

$$\boxed{\lambda_{1,2} = \frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}}$$

The assumed solution $x(t) = \bar{c}e^{-\lambda t}$ becomes

$$\boxed{x(t) = \bar{c}_1 e^{-\lambda_1 t} + \bar{c}_2 e^{-\lambda_2 t} = \bar{c}_1 e^{-\left(\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t} + \bar{c}_2 e^{-\left(\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)t}}$$

From the above equation it can be concludet that:

1. The motion $x(t)$ will be diverging if λ_1 and λ_2 are real and negative (motion = diverging + aperiodic)

2. The motion $x(t)$ will also diverge if λ_1 and λ_2 are complex conjugates with negative real parts.

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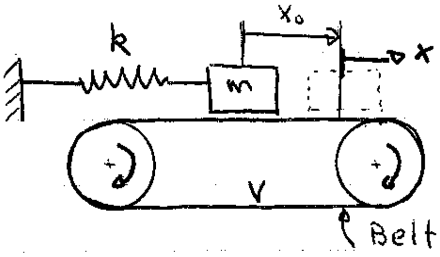
Exercice 14: Instability of a Spring-Supported Mass on Moving Belt.



V = Belt's velocity

Assume friction coefficient between block and belt to be given by:

$$\boxed{\mu = \mu_s - c v}$$

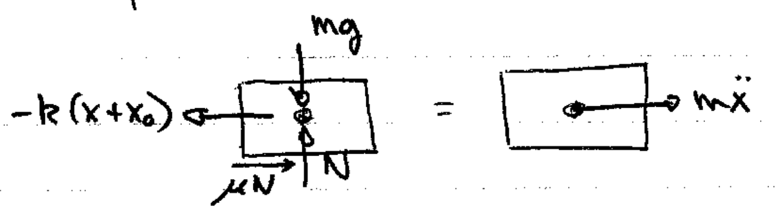
where,

$$\begin{cases} \mu_s \equiv \text{static friction coefficient} \\ c \equiv \text{constant} \\ v \equiv \text{rubbing velocity} \end{cases}$$

- Kinematics: $\boxed{v = V - \dot{x} = (\text{Belt velocity} - \text{Block velocity})}$

- Let us write the equation of motion of the block 'm' from its equilibrium position

$$\sum F_x = \cancel{m\ddot{x}}^{\,0}$$



$$\hookrightarrow \; -k(\cancel{x}^{\,0} + x_0) + \mu N = 0$$

$$\Rightarrow -k x_0 + mg\,\mu = 0$$

$$\Rightarrow -k x_0 + mg\left[\mu_s - c v\right] = 0$$

$$\Rightarrow -k x_0 + mg\left[\mu_s - c(V - \cancel{\dot{x}}^{\,0})\right] \Rightarrow \boxed{x_0 = \frac{mg}{k}\left[\mu_s - cV\right]}$$

The equation of motion for free vibration can be written as,

$$\Sigma F_x = m\ddot{x} \Rightarrow -k(x+x_0) + \mu N = m\ddot{x}$$
$$\Rightarrow -k\left\{ x + \frac{mg}{k}\left[\cancel{\mu_s} - cV\right]\right\} + mg\left[\cancel{\mu_s} - cv\right] = m\ddot{x}$$

$$\Rightarrow -kx + mgcV - mgcv = m\ddot{x}$$
$$\Rightarrow -kx + mgc(V-v) = m\ddot{x} \quad \text{using } v = V - \dot{x} \Rightarrow \dot{x} = V - v$$

So $$\boxed{m\ddot{x} - mgc\dot{x} + kx = 0}$$

where mgc is the damping coefficient resulting from the friction μ being in function of the motion.

The solution to the above differential equation of motion is:

① Assume $\boxed{x(t) = \bar{c}e^{-\lambda t}}$
② Substitute into EOM (Equation of Motion)
$$\bar{c}e^{\lambda t}\left[m\lambda^2 + mgc\lambda + k\right] = 0$$
③ Characteristic Equation
$$\boxed{m\lambda^2 + mgc\lambda + k = 0}$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

where $\begin{cases} b = mgc \\ a = m \\ \Delta = b^2 - 4ac = (mgc)^2 - 4mk \end{cases}$

ME-400

The two roots of the characteristic equation are,

$$\lambda_{1,2} = \frac{-mgc \pm \sqrt{(mgc)^2 - 4mk}}{2m}$$

$$\Rightarrow \boxed{\lambda_{1,2} = -\frac{gc}{2} \pm \sqrt{\left(\frac{gc}{2}\right)^2 - \frac{k}{m}}}$$

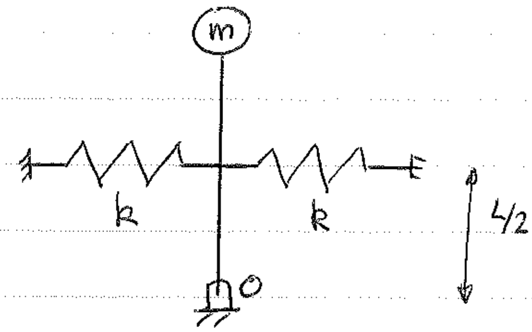
The assumed solution $x(t)$ becomes,

$$x(t) = \bar{c}e^{-\lambda t} = \bar{c}_1 e^{-\left[-\frac{gc}{2} + \sqrt{gc - \frac{4k}{m}}\right]t} + \bar{c}_2 e^{-\left[-\frac{gc}{2} - \sqrt{gc - \frac{4k}{m}}\right]t}$$

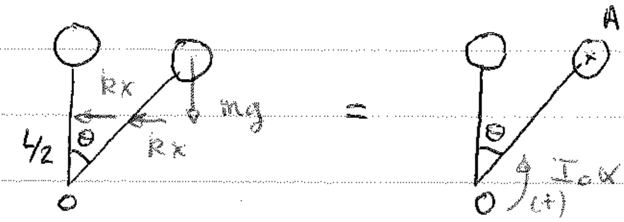
$$\Rightarrow \boxed{x(t) = \underbrace{e^{+\frac{gc}{2}t}}_{\text{term 1}} \left\{ \bar{c}_1 e^{\sqrt{\left(\frac{gc}{2}\right)^2 - \frac{k}{m}}\cdot t} + \bar{c}_2 e^{-\sqrt{\left(\frac{gc}{2}\right)^2 - \frac{k}{m}}\cdot t} \right\}}$$

From **term 1** in the above equation it can be seen that $x(t)$ will increase with time making the system $\boxed{\text{UNSTABLE}}$.

Exercice:



① FBD



② Newton's Eqt

$$\circlearrowright + \sum M_O = I_O\alpha \quad \text{(only rotation about point O)}$$

$$\Rightarrow -mgL\sin\theta + 2kx\frac{L}{2}\cos\theta = I_O\alpha \quad — ①$$

③ Kinematics

From the figure $x = \frac{L}{2}\sin\theta$

$$\bar{a}_A = \cancel{\bar{a}_O}^{\,0} + \bar{\alpha}\times\bar{R}_{A/O} - \cancel{\omega^2\bar{R}_{A/O}}^{\,\omega=0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \alpha \\ R\sin\theta & R\cos\theta & 0 \end{vmatrix} \Rightarrow \bar{a}_A = R\alpha\begin{pmatrix}-\cos\theta \\ \sin\theta\end{pmatrix}$$

$\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$

The vector $\begin{pmatrix}-\cos\theta \\ \sin\theta\end{pmatrix}$ indicates a positive rotation going counter-clockwise. However, on the figure we assumed a clockwise rotation. As a result eqt ① becomes

$$-mgL\sin\theta + k\frac{L}{2}\sin\theta\cdot\cos\theta = -I_O\ddot{\theta}$$

For small angles $\begin{cases}\sin\theta \simeq \theta \\ \cos\theta \simeq 1\end{cases}$ $\quad \theta < \pi/20$

The equation of motion becomes:

$$\frac{kL^2}{2}\theta - mgL\,\theta = -I_o\ddot{\theta}$$

$$\Rightarrow \quad I_o\ddot{\theta} + \left(\frac{kL^2}{2} - mgL\right)\theta = 0$$

If $I_o = mL^2$, the equation of motion can be rewritten as,

$$\boxed{\ddot{\theta} + \omega_\theta^2\,\theta = 0}$$

where $\omega_\theta^2 = \dfrac{kL^2/2 - mgL}{mL^2}$, sign depends on $\dfrac{kL^2}{2} - mgL$

③ Solve

Ⓘ Assume $\theta = \theta_0 e^{-\lambda t}$

$$\begin{cases} \dot{\theta} = -\theta_0 \lambda e^{-\lambda t} \\ \ddot{\theta} = \theta_0 \lambda^2 e^{-\lambda t} \end{cases}$$

Ⓘⓘ Substitute into $\ddot{\theta} + \omega_\theta^2\theta = 0$

$$\ddot{\theta} + \omega_\theta^2\theta = 0 \Rightarrow \theta_0 e^{-\lambda t}\left[\lambda^2 + \omega_\theta^2\right] = 0$$

The above equation is satisfied if and only if

$$\begin{cases} \theta_0 e^{-\lambda t} = 0 & \text{(Trivial Solution)} \\ \lambda^2 + \omega_\theta^2 = 0 & \Rightarrow \lambda = \pm i \cdot \omega_\theta \end{cases}$$

So that the assumed solution becomes:

$$\boxed{\theta(t) = \bar{c}_1 \cdot e^{-\lambda_1 t} + \bar{c}_2 e^{-\lambda_2 t}}$$

OR $$\boxed{\theta(t) = \bar{c}_1 e^{-i\omega_0 t} + \bar{c}_2 e^{i\omega_0 t}}$$

If $\omega_0 < 0 \Rightarrow \theta(t) = c_1 e^{\omega_0 t} + c_2 e^{-\omega_0 t}$

$$\left.\begin{array}{l} \lim_{t\to\infty} c_1 e^{\omega_0 t} = \infty \\ \lim_{t\to\infty} c_2 e^{-\omega_0 t} = 0 \end{array}\right\} \Rightarrow \lim_{t\to\infty} \theta(t) = \infty$$

↳ the solution diverges as t increase ⇒ UNSTABLE.

If $\omega_0 > 0 \Rightarrow \theta(t) = \bar{c}_1 e^{-i\omega_0 t} + \bar{c}_2 e^{i\omega_0 t}$

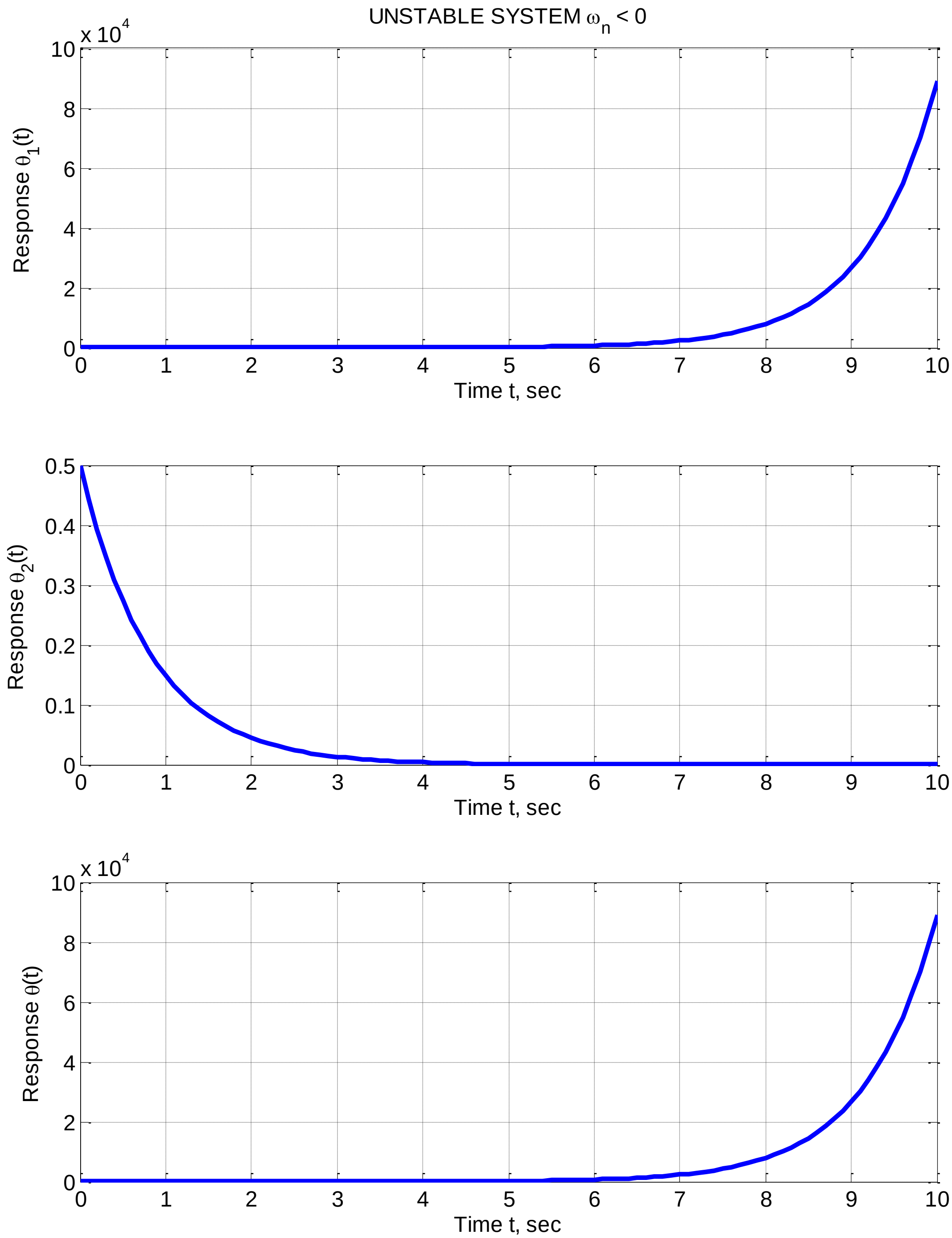
When plotting complex functions in vibrations only the real part is of interest. Recalling the Euler's relations

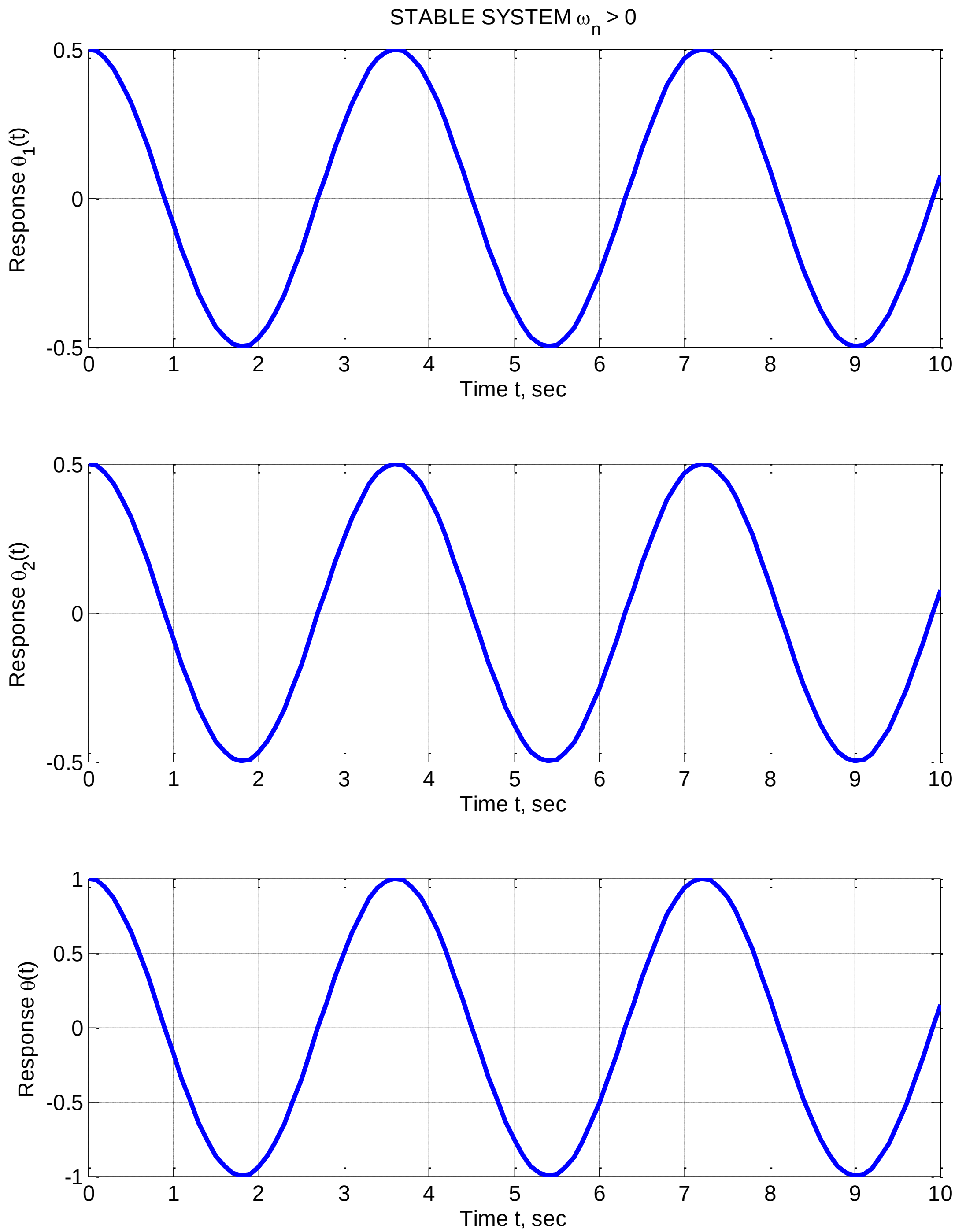
$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

↳

$$\theta(t) = \underbrace{(\bar{c}_1 + \bar{c}_2)\cos\theta}_{\text{Real}} + \underbrace{(\bar{c}_1 - c_2)\cdot\sin\theta\cdot i}_{\text{Imaginary}}$$

So when $\omega_0 > 0$, the response $\theta(t)$ is harmonic with amplitude $\bar{c}_1 + \bar{c}_2$. ⇒ STABLE


UNSTABLE SYSTEM $\omega_n < 0$
x 10^4
Response $\theta_1(t)$
Time t, sec
Response $\theta_2(t)$
Time t, sec
x 10^4
Response $\theta(t)$
Time t, sec



STABLE SYSTEM $\omega_n > 0$
Response $\theta_1(t)$
0.5
0
-0.5
0 1 2 3 4 5 6 7 8 9 10
Time t, sec
Response $\theta_2(t)$
0.5
0
-0.5
0 1 2 3 4 5 6 7 8 9 10
Time t, sec
Response $\theta(t)$
1
0.5
0
-0.5
-1
0 1 2 3 4 5 6 7 8 9 10
Time t, sec

MATLAB

```
clear all
clc
M=[10,100];
k=100;
g=9.81;
L=5;
c1=0.5;      % initial conditions
c2=c1;
t=0:0.1:10;

%=================================================================
===========
for j=1:2
    m=M(j);
    omega = sqrt(((1/2)*k*L^2-m*g*L)/(m*L^2));
    term=((1/2)*k*L^2-m*g*L)/(m*L^2);
    lambda_1 = i*omega;
    lambda_2 = -i*omega;

    theta = c1*exp(-lambda_1*t)+c2*exp(-lambda_2*t);
    %theta=c1*exp(-i*sqrt(((1/2)*k*L^2-m*g*L)/(m*L^2))*t)...
    %      +c2*exp(i*sqrt(((1/2)*k*L^2-m*g*L)/(m*L^2))*t);
    theta_1 = c1*exp(-lambda_1*t);
    %theta_1=c1*exp(-i*sqrt(((1/2)*k*L^2-m*g*L)/(m*L^2))*t);
    theta_2 = c2*exp(-lambda_2*t);
    %theta_2 = c2*exp(i*sqrt(((1/2)*k*L^2-m*g*L)/(m*L^2))*t);
    if term < 0
        fprintf('\n UNSTABLE SYSTEM\n')
        figure(1)
        subplot(311)
        plot(t, theta_1)
        xlabel('Time t, sec')
        ylabel('Response \theta_1(t)')
        grid on
        title('UNSTABLE SYSTEM \omega_n < 0')
        subplot(312)
        plot(t, theta_2)
        xlabel('Time t, sec')
        ylabel('Response \theta_2(t)')
        grid on
        subplot(313)
        plot(t, theta)
        xlabel('Time t, sec')
        ylabel('Response \theta(t)')
        grid on
```

```
    elseif term >0
        fprintf('\n STABLE SYSTEM\n')
        figure(2)
        subplot(311)
        plot(t, theta_1)
        xlabel('Time t, sec')
        ylabel('Response \theta_1(t)')
        grid on
        title('STABLE SYSTEM \omega_n > 0')
        subplot(312)
        plot(t, theta_2)
        xlabel('Time t, sec')
        ylabel('Response \theta_2(t)')
        grid on
        subplot(313)
        plot(t, theta)
        xlabel('Time t, sec')
        ylabel('Response \theta(t)')
        grid on
    end
end
```

MAPLE

> *restart*;
>

INVERTED PENDULUM STABILITY PROBLEM

> theta := $c1\cdot\exp(-\lambda[1]\cdot t) + c2\cdot\exp(-\lambda[2]\cdot t)$;

$$\theta := c1\, e^{-\lambda_1 t} + c2\, e^{-\lambda_2 t}$$

> omega := $\mathrm{sqrt}\left(\frac{\left(\frac{kL^2}{2} - m\cdot g\cdot L\right)}{m\cdot L^2}\right)$; $\lambda[1]$:= omega·I; lambda[2] := −omega·I;

$$\omega := \sqrt{\frac{\frac{1}{2}\, kL^2 - m\, g\, L}{m\, L^2}}$$

$$\lambda_1 := \mathrm{I}\sqrt{\frac{\frac{1}{2}\, kL^2 - m\, g\, L}{m\, L^2}}$$

$$\lambda_2 := -\mathrm{I}\sqrt{\frac{\frac{1}{2}\, kL^2 - m\, g\, L}{m\, L^2}}$$

> theta;

$$c1\, e^{-\mathrm{I}\sqrt{\frac{\frac{1}{2}\, kL^2 - m\, g\, L}{m\, L^2}}\, t} + c2\, e^{\mathrm{I}\sqrt{\frac{\frac{1}{2}\, kL^2 - m\, g\, L}{m\, L^2}}\, t}$$

> *restart*;

> m := 10; k := 100; g := 9.81; L := 5; omega := $\mathrm{sqrt}\left(\frac{\left(\frac{k\cdot L^2}{2} - m\cdot g\cdot L\right)}{m\cdot L^2}\right)$;

$$m := 10$$

$$k := 100$$

$$g := 9.81$$

$$L := 5$$

$$\omega := 1.742985944$$

> theta := $c1\cdot\exp(-\lambda[1]\cdot t) + c2\cdot\exp(-\lambda[2]\cdot t)$;

$$\theta := c1\, e^{-\lambda_1 t} + c2\, e^{-\lambda_2 t}$$

> $\lambda[1]$:= omega·I; lambda[2] := −omega·I;

$$\lambda_1 := 1.742985944\,\mathrm{I}$$

$$\lambda_2 := -1.742985944\,\mathrm{I}$$

> theta;

$$c1\, e^{-1.742985944\,\mathrm{I}\, t} + c2\, e^{1.742985944\,\mathrm{I}\, t}$$

> *restart*;

> m := 100; k := 100; g := 9.81; L := 5; omega := $\mathrm{sqrt}\left(\frac{\left(\frac{k\cdot L^2}{2} - m\cdot g\cdot L\right)}{m\cdot L^2}\right)$;

$$m := 100$$

$$k := 100$$

$$g := 9.81$$

$$L := 5$$

$$\omega := 1.209131920\,\mathrm{I}$$

> theta := $c1\cdot\exp(-\lambda[1]\cdot t) + c2\cdot\exp(-\lambda[2]\cdot t)$;

$$\theta := c1\,\mathrm{e}^{-\lambda_1 t} + c2\,\mathrm{e}^{-\lambda_2 t}$$

> λ[1] := omega·*I*; lambda[2] := −omega·*I*;

$$\lambda_1 := -1.209131920$$

$$\lambda_2 := 1.209131920$$

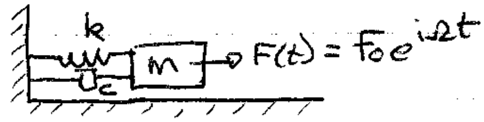
> theta;

$$c1\,\mathrm{e}^{1.209131920\,t} + c2\,\mathrm{e}^{-1.209131920\,t}$$

>

12.0 Energy Dissipation Due to Hysteresis Damping

Consider a spring-mass-damper subjected to a harmonic excitation (see section 11.1)



$$\boxed{m\ddot{x} + c\dot{x} + kx = F_0 e^{i\Omega t}}$$

Assuming the solution of the form $\boxed{x(t) = X_0 e^{i\Omega t}}$, the solution was found to be

$$\boxed{|X_0| = \frac{F_0}{m} H_m(\Omega)}$$

where $$H_m(\Omega) = \frac{1}{\omega_n^2\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

- Because of the damping force, the system is clearly not conservative, and indeed energy is dissipated. Since the energy dissipation must be equal to the work done by the external force, the expression for the energy dissipated per cycle can be writen as:

$$\Delta E_{cycle} = \int_{cycle} F\,dx = \int_0^{2\pi/\Omega} F \cdot \dot{x}\,dt \qquad \dot{x} = \frac{dx}{dt} \Rightarrow dx = \dot{x}\,dt$$

Considering only the real part of $x(t) = X_0 e^{i\Omega t} = X_0 \cos(\Omega t - \phi)$ (Phase diff. ϕ)

$F(t) = F_0 e^{i\Omega t} = F_0 \cos(\Omega t)$

$$\rightarrow \Delta E_{cycle} = \int_0^{2\pi/\Omega} F \cdot \dot{x}\,dt = \int_0^{2\pi/\Omega} F_0 \cos(\Omega t)(-X_0 \Omega)\sin(\Omega t - \phi)\,dt$$

(continues next page ...)

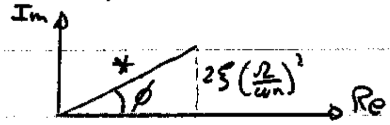
$$\Delta E_{cycle} = -F_o X_o \Omega \int_0^{2\pi/\Omega} \cos(\Omega t)\cdot\sin(\Omega t - \phi)\, dt$$

$$\Rightarrow \boxed{\Delta E_{cycle} = \Omega F_o X_o \pi \sin\phi}$$ (see attached maple file)

where

$$\sin\phi = \frac{2\xi\left(\frac{\Omega}{\omega_n}\right)}{\underbrace{\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}_{*}}$$

see p: 58 +



Substituting into ΔE_{cycle},

$$\rightarrow \Delta E_{cycle} = \Omega F_o X_o \pi \frac{2\xi\left(\frac{\Omega}{\omega_n}\right)}{\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

Using $\boxed{2\xi = \frac{c}{m\omega_n}}$

$$\rightarrow \Delta E_{cycle} = \Omega F_o X_o \pi \left(\frac{c}{m}\right)\left(\frac{\Omega}{\omega_n^2}\right) \frac{1}{\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

OR $\boxed{X_o = \frac{F_o}{m} H_m(\Omega)}$

where $$H_m(\Omega) = \frac{1}{\omega_n^2\left[\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + 4\xi^2\left(\frac{\Omega}{\omega_n}\right)^2\right]^{1/2}}$$

$$\rightarrow \boxed{\Delta E_{cycle} = X_o^2 \pi c \Omega} \quad -(i)$$

(continues next page...)

```
> restart;
> x:=X[0]*cos(Omega*t-phi);
```

$$x := X_0 \cos(\Omega\, t - \phi)$$

```
> v:=diff(x,t);
```

$$v := -X_0 \sin(\Omega\, t - \phi)\, \Omega$$

```
> f:=F[0]*cos(Omega*t);
```

$$f := F_0 \cos(\Omega\, t)$$

```
> eq:=simplify(f*v);
```

$$eq := -F_0 \cos(\Omega\, t)\, X_0 \sin(\Omega\, t - \phi)\, \Omega$$

```
> int(eq,t);
```

$$F_0\, X_0\, \Omega \left(\frac{1}{2} \sin(\phi)\, t + \frac{1}{4} \frac{\cos(2\, \Omega\, t - \phi)}{\Omega} \right)$$

```
> Exp:=int(eq,t=0..2*pi/Omega);
```

$$Exp := F_0\, X_0 \sin(\phi)\, \pi + 2\, F_0\, X_0 \cos(\phi) \cos(\pi)^4 - 2\, F_0\, X_0 \cos(\phi) \cos(\pi)^2 + 2\, F_0\, X_0 \sin(\phi) \sin(\pi) \cos(\pi)^3 - F_0\, X_0 \sin(\phi) \sin(\pi) \cos(\pi)$$

```
> cos(pi):=1;sin(pi):=0;
```

$$\cos(\pi) := 1$$

$$\sin(\pi) := 0$$

```
> Exp;
```

$$F_0\, X_0 \sin(\phi)\, \pi$$

- Experience shows that energy is dissipated in all real systems, including those systems for which the mathematical model makes no specific provision for damping (Spring-Mass). For instance, energy is always dissipated in spring as a resulting of the internal friction, a.k.a. hysteresis or structural damping.

- Experiments performed on a large variety of materials have shown that the energy lost per cycle is roughly proportional to the square of the displacement amplitude.

$$\boxed{\Delta E_{cycle} = \alpha X_0^2} \quad \text{-(ii)}$$

where α is a constant independent of Ω.

- Equating (i) and (ii),

$$\Delta E_{cycle} = \cancel{X_0^2} \pi c \Omega = \alpha \cancel{X_0^2}$$

$$\Rightarrow \boxed{c = \frac{\alpha}{\pi \Omega}}$$

only valid for $F(t) = F_0 e^{i\Omega t}$ (important)

This enables to write the equation of motion of a system due to internal friction as

$$\boxed{m\ddot{x} + \frac{\alpha}{\pi\Omega}\dot{x} + kx = F_0 e^{i\Omega t}}$$