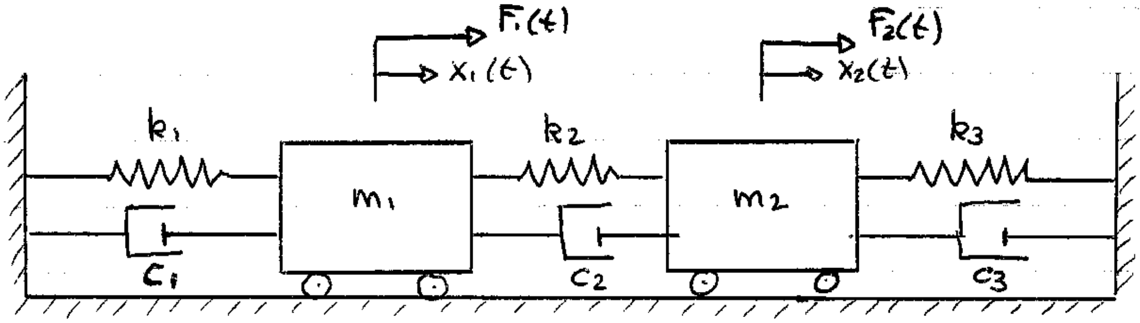
Chap 2 Two and Multi-Degree of Freedom (MDOF) systems

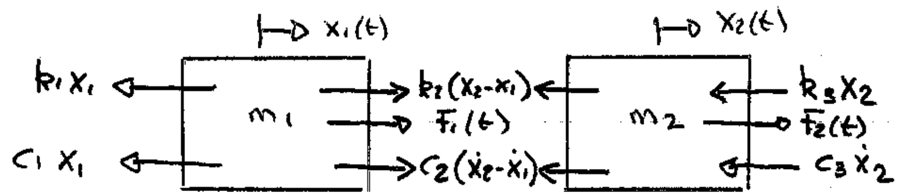
1. Equations of Motion for Forced-Vibration

The viscously damped system shown below represents a classic 2-degree of freedom system



The equations of motion for the two-degree of freedom are derived next,

Free-Body-Diagram (FBD):



Motion ⇒ Translation ⇒ $\Sigma \bar{F} = m\bar{a}$

- For block of mass m_1: $\Sigma F_x = m_1 \ddot{x}_1$

$$F_1(t) - k_1 x_1 + k_2(x_2 - x_1) - c_1 \dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) = m_1 \ddot{x}_1$$

$$\Rightarrow m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = F_1(t) \quad (i)$$

For block of mass m_2: $\Sigma F_x = m_2 \ddot{x}_2$

$$\Rightarrow F_2(t) - k_3 x_2 - k_2(x_2 - x_1) - c_3 \dot{x}_2 - c_2(\dot{x}_2 - \dot{x}_1) = m_2 \ddot{x}_2$$

$$\Rightarrow m_2 \ddot{x}_2 + c_3 \dot{x}_2 + k_3 x_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = F_2(t) \quad \text{— (ii)}$$

The two equations of motions are

$$\begin{cases} m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) = F_1(t) \\ m_2 \ddot{x}_2 + c_3 \dot{x}_2 + k_3 x_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = F_2(t) \end{cases}$$

OR

$$\begin{cases} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 = F_1(t) \\ m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 + (c_2 + c_3) \dot{x}_2 - c_2 \dot{x}_1 = F_2(t) \end{cases}$$

In matrix form:

$$\boxed{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}}$$

2. Free Vibration Analysis of Undamped System

The system of equations of motion derived in the previous section reduces to:

$$\boxed{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Assuming Harmonic solution $\boxed{x(t) = \mathbf{X} e^{i\lambda t}}$

Substituting the harmonic solution into the equations of motion

$$\Rightarrow \quad e^{i\lambda t}\left\{\begin{bmatrix} -m_1\lambda^2+(k_1+k_2) & -k_2 \\ -k_2 & -m_2\lambda^2+(k_2+k_3)\end{bmatrix}\begin{Bmatrix}X_1\\X_2\end{Bmatrix}\right\}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

The non-trivial solution of the above equation exists only when,

$$\det\left\{\begin{bmatrix} -m_1\lambda^2+(k_1+k_2) & -k_2 \\ -k_2 & -m_2\lambda^2+(k_2+k_3)\end{bmatrix}\right\}=0$$

$$\Rightarrow \quad \boxed{(m_1m_2)\lambda^4-\{(k_1+k_2)m_2+(k_2+k_3)m_1\}\lambda^2+\{(k_1+k_2)(k_2+k_3)-k_2^2\}=0}$$

The above equation is known as the characteristic equation. Solution of the characteristic equation yields to the eigenvalues of the system.

$$\lambda_1^2,\lambda_2^2=\frac{1}{2}\left\{(k_1+k_2)m_2+(k_2+k_3)m_1\right\}$$
$$\mp\frac{1}{2}\left[\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right\}^2\cdots -4\left\{\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1m_2}\right\}\right]^{1/2}$$

The eigenvalues or characteristic-values correspond to the natural frequencies of the system.

$$\boxed{\lambda^2=\omega_n^2}$$

- The values of $\bar{X}_1$ and $\bar{X}_2$ remain to be determined. These values depend on the eigenvalues λ_1^2 and λ_2^2.

1st Case : $\lambda = \lambda_1$

$$e^{i\lambda_1 t}\left\{\begin{bmatrix} -m_1\lambda_1^2+(k_1+k_2) & -k_2 \\ -k_2 & -m_2\lambda_1^2+(k_2+k_3)\end{bmatrix}\begin{Bmatrix}\bar{X}_1 \\ \bar{X}_2\end{Bmatrix}\right\} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

Since the equation must be satisfied at any 't', the term within brackets must be zero

$$\Rightarrow \{-m_1\lambda_1^2+(k_1+k_2)\}\bar{X}_1 - k_2\bar{X}_2 = 0$$

$$-k_2\bar{X}_1 + \{-m_2\lambda_1^2+(k_2+k_3)\}\bar{X}_2 = 0$$

$$\Rightarrow \boxed{\frac{\bar{X}_2^{(1)}}{\bar{X}_1^{(1)}} = \frac{-m_1\lambda_1^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\lambda_1^2+(k_2+k_3)}}$$

2nd Case $\lambda = \lambda_2$

Following the same procedure as for the 1st case,

$$\boxed{\frac{\bar{X}_2^{(2)}}{\bar{X}_1^{(2)}} = \frac{-m_1\lambda_2^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\lambda_2^2+(k_2+k_3)}}$$

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ME-400

The eigenvectors or normal-modes of vibration correspon-ding to λ_1^2 and λ_2^2 can be expressed, respectively

$$\bar{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ \frac{X_2^{(1)}}{X_1^{(1)}} \cdot X_1^{(1)} \end{Bmatrix}$$

and

$$\bar{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ \frac{X_2^{(2)}}{X_1^{(2)}} \cdot X_1^{(2)} \end{Bmatrix}$$

The vectors $\bar{X}^{(1)}$ and $\bar{X}^{(2)}$ denotes the normal-modes of vibration which are also known as modal vectors.

- The free-vibration solutions in time are:

<u>1st mode</u>: $$x^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} e^{i\lambda_1 t} \\ \frac{X_2^{(1)}}{X_1^{(1)}} \cdot X_1^{(1)} e^{i\lambda_1 t} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \sin(\omega_1 t - \phi_1) \\ \frac{X_2^{(1)}}{X_1^{(1)}} \cdot X_1^{(1)} \cdot \sin(\omega_1 t - \phi) \end{Bmatrix}$$

<u>2nd mode</u>: $$x^{(2)}(t) = \begin{Bmatrix} x_2^{(1)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} e^{i\lambda_2 t} \\ \frac{X_2^{(2)}}{X_1^{(2)}} X_1^{(2)} e^{i\lambda_2 t} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \sin(\omega_2 t - \phi_2) \\ \frac{X_2^{(2)}}{X_1^{(2)}} \cdot X_1^{(2)} \sin(\omega_2 t - \phi_2) \end{Bmatrix}$$

where the constants $X_1^{(1)}, X_1^{(2)}, \phi_1 \,\&\, \phi_2$ are determined from the initial conditions.

- Two initial-conditions for each mass are required to determined the constants. In the special case

$$\begin{cases} x_1(t=0) = X_1^{(1)} = \text{constant}, & \dot{x}_1(t=0) = 0 \\ x_2(t=0) = \frac{X_2^{(2)}}{X_1^{(2)}} X_1^{(2)}, & \dot{x}_2(t=0) = 0 \end{cases}$$

However, for any other type of initial conditions both modes would be excited. The general solution representing the resulting motion is obtained by superposition

$$\boxed{x(t) = \bar{x}_1(t) + \bar{x}_2(t)}$$

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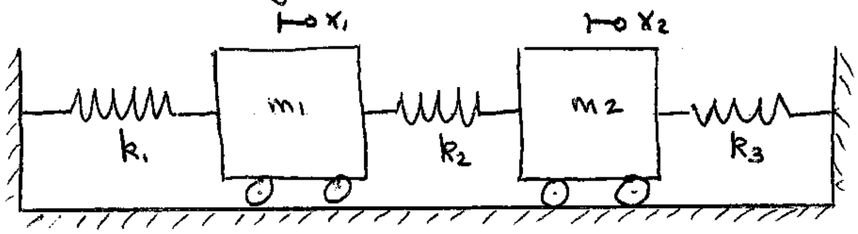
Exercice 1 :

Consider the system below,



$m_1 = m \qquad k_1 = k_2 = k \qquad m = k = 1$ for MATLAB code.

$m_2 = 2m \qquad k_3 = 2k$

Determine the eigen-values and eigenvectors

The system of equations of motion in matrix form becomes

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

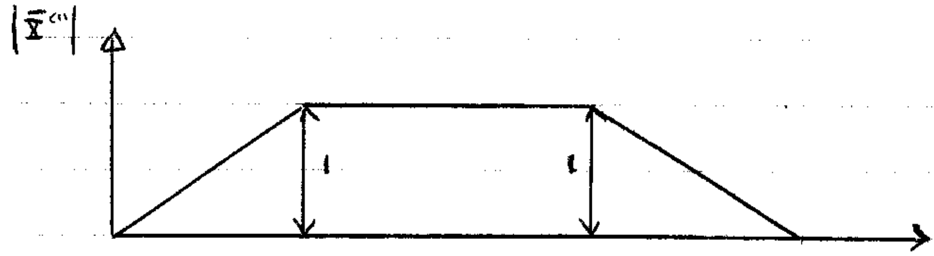
- Results using the formulation in pages 3-5

$$\boxed{\begin{cases} \lambda_1^2 = 1 \\ \lambda_2^2 = 2.5 \end{cases}}$$

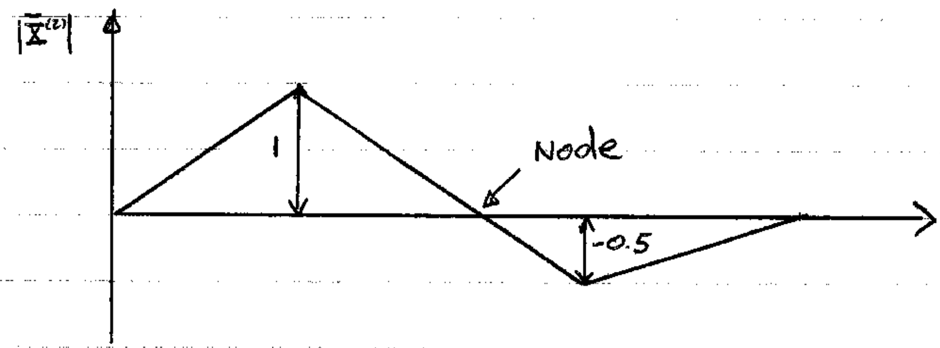
$$\frac{\bar{X}_2^{(1)}}{\bar{X}_1^{(1)}} = 1, \text{ Assuming } \bar{X}_1^{(1)} = 1 \Rightarrow \boxed{\bar{X}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\frac{\bar{X}_2^{(2)}}{\bar{X}_1^{(2)}} = -0.5, \text{ Assuming } \bar{X}_1^{(2)} = 1 \Rightarrow \boxed{\bar{X}^{(2)} = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}}$$

Mode-Vector 1, $\bar{X}^{(1)}$



Mode-Vector 2, $\bar{X}^{(2)}$



Note: A point of zero displacement is called a node

- Results using MATLAB:

$$\boxed{\begin{cases} \lambda_1^2 = 1 \\ \lambda_2^2 = 2.5 \end{cases}}$$

$$\bar{X}^{(1)} = \begin{pmatrix} -0.5774 \\ -0.5774 \end{pmatrix}$$

$$\bar{X}^{(2)} = \begin{pmatrix} -0.8165 \\ 0.4082 \end{pmatrix}$$

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It is observed that a first glance the eigen-vectors obtained using MATLAB are different than the obtained using the classnotes formulation. The difference is due to the assumed $X_1^{(1)}=1$ and $X_1^{(2)}=1$ in the classnote formulation. It can be easily seen than between the MATLAB Results and classnotes results there is only a difference of a constant. (Mode normilization)

$$\bar{X}^{(1)} = \begin{pmatrix} -0.5774 \\ -0.5774 \end{pmatrix} \quad \text{if} \quad c_1 = (-0.5774)^{-1}$$

$$\Rightarrow \quad \bar{X}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{X}^{(2)} = \begin{pmatrix} -0.86163 \\ 0.4082 \end{pmatrix} \quad \text{if} \quad c_2 = (-0.86165)^{-1}$$

$$\Rightarrow \quad \bar{X}^{(2)} = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}$$

The "shape" of the eigen-vector is not affected by the arbitrary values assumed $X_1^{(1)}$ and $X_2^{(2)}$. In order to "match" results it is sufficient to normalize the eigen-vectors (mode-shapes) with respect to a constant.

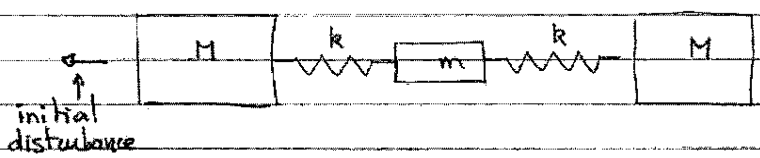
MATLAB INPUT FILE

```
% Example 1
% Two-Degree of Freedom
clear all;
clc;

k=1;
m=1;

m1=m;
m2=2*m;
k1=k;
k2=k1;
k3=2*k;

% Using Classnotes Formulation
fprintf('Using Class-Notes Formulation...\n')
term1=((k1+k2)*m2+(k2+k3)*m1)/m1/m2;
term2=term1;
term3=((k1+k2)*(k2+k3)-k2^2)/m1/m2;

% Eigen-values
r1=1/2*term1-1/2*sqrt(term2^2-4*term3);
r2=1/2*term1+1/2*sqrt(term2^2-4*term3);
fprintf('\n Eigen-Value 1 = %g \n', r1)
fprintf('\n Eigen-Value 2 = %g \n', r2)

% Eigen-Vectors
X2_X1_mode1=(-m1*r1+(k1+k2))/k2;
X2_X1_mode2=(-m1*r2+(k1+k2))/k2;

% Normal-Modes or Modal Vectors
fprintf('\n Eigen-Vectors \n')
% Modal Vector for Mode 1
%Assume X1=1
X1=1;
mode1=[X1 ; X2_X1_mode1*X1]
% Modal Vector for Mode 2
%Assume X1=1
X1=1;
mode2=[X1 ; X2_X1_mode2*X1]

fprintf('==============================================================
== \n')
% Using Matlab and Matrix Notation
fprintf('Using Matlab and Matrix Formulation...\n')
M=[m1 0;0 m2];
K=[k1+k2 -k2;-k2 k2+k3];
% function eig in Matlab to eigen-Values/Vectors
[Eigen_vectors,Eigen_values]=eig(K,M);

Eg_values=diag(Eigen_values);
fprintf('\n Eigen-Value 1 = %g \n', Eg_values(1))
fprintf('\n Eigen-Value 2 = %g \n', Eg_values(2))
```

```
fprintf('\n Eigen-Vectors \n')
Eigen_vectors(:,1)
Eigen_vectors(:,2)


%Normalize to 1
fprintf('\n Normalization of Eigen-Vectors to 1... \n')
for i=1:2
    work1=max(Eigen_vectors(:,i));
    work2=min(Eigen_vectors(:,i));
    if abs(work1)>=abs(work2);
    New_Eig_Vector(:,i)=Eigen_vectors(:,i)/work1;
    else abs(work1)<abs(work2);
    New_Eig_Vector(:,i)=Eigen_vectors(:,i)/work2;
    end
end

fprintf('\n Normalized Eigen-Vectors \n')
New_Eig_Vector(:,1)
New_Eig_Vector(:,2)


fprintf('==============================================================
== \n')

fprintf('\n Check Orthogonality of Mode Vectors \n')
fprintf('Using Class-Notes Formulation...\n')
Phi=[mode1 mode2];
Phi'*M*Phi
Phi'*K*Phi

fprintf('Using Matlab and Matrix Formulation...\n')
Phi=New_Eig_Vector;
Phi'*M*Phi
Phi'*K*Phi

fprintf('\n Note:')
fprintf('\n The orthogonal property of the mode shapes is very
important since it')
fprintf('\n decouples the equation of motion.  Decoupling means that
the first')
fprintf('\n equation of motion will be in function of x1 only and the
second equation of motion')
fprintf('\n in function of x2 only.  In other words the 2 degree  of
freedom systen is')
fprintf('\n reduced to two 1 single degree of freedom systems')
```

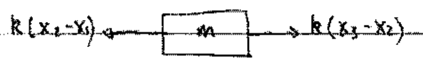
Extra - Problem

Eigenvalues = Natural/Resonance Frequencies
Eigenvectors = Natural Modes / Mode Shapes



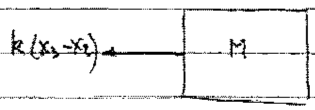
, $k(x_2 - x_1) = M\ddot{x}_1$

$\Rightarrow -kx_1 + kx_2 = M\ddot{x}_1$ — ①



, $-k(x_2 - x_1) + k(x_3 - x_2) = m\ddot{x}_2$

$\Rightarrow kx_1 - 2kx_2 + kx_3 = m\ddot{x}_2$ — ②



, $-k(x_3 - x_2) = M\ddot{x}_3$

$\Rightarrow kx_2 - kx_3 = M\ddot{x}_3$ — ③

Combining ①, ② and ③ and using the matrix notation:

$$\begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \{0\}$$

Assume solution of the form: $\boxed{\{x(t)\}_n = \{X\}_n e^{i\omega t}}$

Extra-Problem (cont)

Substitute into the eqt. of motion (*)

$$\left(\begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} - \omega_n^2 \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix}\right)\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} e^{i\omega t} = 0$$

OR

$$\boxed{\begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega_n^2 \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}} \qquad (1)$$

Which characteristic eqt. is

$$\det\begin{pmatrix} -k-M\omega_n^2 & k & 0 \\ k & -2k-m\omega_n^2 & k \\ 0 & k & -k-M\omega_n^2 \end{pmatrix} = 0$$

$$\Rightarrow -(k+\omega_n^2 M)\left[+(2k+\omega_n^2 m)(k+\omega_n^2 M)-k^2\right]+k\left(k+\omega_n^2 M\right)=0$$

$$\Rightarrow -(k+\omega_n^2 M)\left[2k^2+2k\omega_n^2 M+k\omega_n^2 m+\omega_n^4 mM-k^2-k^2\right]$$

$$\Rightarrow -(k+\omega_n^2 M)\left[\omega_n^2\left(2kM+km+\omega_n^2 mM\right)\right]$$

OR $\boxed{-\omega_n^2(k+\omega_n^2 M)(\omega_n^2 mM+km+2kM)=0}$

Extra - Problem (cont)

Solving the previous eqt for ω_n^2, we get:

$$\boxed{\begin{aligned} &\omega_1^2 = 0 \\ &\omega_2^2 = \frac{k}{M} \qquad \text{Eigenvalues (Natural / Resonance Frequencies)} \\ &\omega_3^2 = \frac{k}{M} + \frac{2k}{m} \end{aligned}}$$

The corresponding eigenvectors are determined by substituting the eigenvalues back into (i) one at a time.

- For $\omega_1^2 = 0$

$$\begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \omega_1^2 \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \{0\}$$

$$\hookrightarrow \left\{ \begin{aligned} -kx_1 + kx_2 + 0 &= 0 \\ kx_1 - 2kx_2 + kx_3 &= 0 \\ 0 + kx_2 - kx_3 &= 0 \end{aligned} \right. \Rightarrow \left\{ \boxed{x_1 = x_2 = x_3} \right.$$

That describes pure translation with no relative motion of the masses and no vibration. $\omega_n^2 = 0 \Rightarrow$ Rigid Body Mode / Motion

Extra - Problem (cont)

- For $\omega_2^2 = -\frac{k}{M}$

$$\begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = -\frac{k}{M} \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{m}{M} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\hookrightarrow \begin{cases} -x_1 + x_2 = -x_1 \\ x_1 - 2x_2 + x_3 = -x_2 \frac{m}{M} \\ x_2 - x_3 = -x_3 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_2\left(2 + \frac{m}{M}\right) = x_1 + x_3 \\ x_2 = 0 \end{cases}$$

$$\hookrightarrow \boxed{\begin{matrix} x_2 = 0 \\ x_1 = -x_3 \end{matrix}}$$

The two outer masses are moving in opposite direction The center mass is stationary.

Extra - Problem (cont)

- For $\omega_3 = -\frac{k}{M} - \frac{2k}{m}$

$$\begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = -\left(\frac{k}{M} + \frac{2k}{m}\right) \begin{bmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\hookrightarrow \begin{cases} -x_1 + x_2 = -(1 + 2M/m)\, x_1 \\ x_1 - 2x_2 + x_3 = -(m/M + 2)\, x_2 \\ x_2 - x_3 = -(1 + 2M/m)\, x_3 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = -\frac{2M}{m} x_1 \\ x_1 - x_3 = -m/M\, x_2 \\ x_2 = -\frac{2M}{m} x_3 \end{cases}$$

$\hookrightarrow \quad x_1 = x_3$

then $\quad x_2 = -\frac{2M}{m} x_1$

The two outer masses are moving together. The center mass is moving opposite to the two outer ones.

⚠ check: In all cases Linear Momentum **must** be zero.

4. Forced Vibration Response

The most general two degree of freedom system, independely of the coordinate axis chosen, is of the form

$$\left[\begin{array}{c|c} m_{11} & m_{12} \\ \hline m_{12} & m_{22} \end{array}\right]\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \left[\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{12} & c_{22} \end{array}\right]\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{12} & k_{22} \end{array}\right]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Solution of the above system is obtained by following the standard procedure described:

1. Assuming harmonic solution

$$x_j = X_j e^{i\Omega t} \Rightarrow \begin{cases} x_1 = X_1 e^{i\Omega t} \\ x_2 = X_2 e^{i\Omega t} \end{cases} \qquad \Omega \equiv \text{forcing frequency}$$

$$f_j = F_j e^{i\Omega t} \Rightarrow \begin{cases} f_1 = F_1 e^{i\Omega t} \\ f_2 = F_2 e^{i\Omega t} \end{cases}$$

2. Substituting into the equation of motion

$$\cancel{e^{i\Omega t}}\left[\begin{array}{c|c} -m_{11}\Omega^2 + ic_{11}\Omega + k_{11} & -m_{12}\Omega^2 + ic_{12}\Omega + k_{12} \\ \hline -m_{12}\Omega^2 + ic_{12}\Omega + k_{12} & -m_{22}\Omega^2 + ic_{22}\Omega + k_{22} \end{array}\right]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \cancel{e^{i\Omega t}}\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Using impedance notation,

$$\boxed{\begin{aligned} &Z_{rs}(i\Omega) = -m_{rs}\Omega^2 + ic_{rs}\Omega + k_{rs} \qquad (r,s = 1,2) \\ &\text{OR} \\ &Z_{rs}(i\Omega) = m_{rs}\omega_{rs}^2\left[\left(1-\left(\frac{\Omega}{\omega_{rs}}\right)^2\right) + i\left(2\xi_{rs}\left(\frac{\Omega}{\omega_{rs}}\right)\right)\right] \end{aligned}}$$

Then the expression,

$$\begin{bmatrix} -m_{11}\Omega^2 + ic_{11}\Omega + k_{11} & -m_{12}\Omega^2 + ic_{12}\Omega + k_{12} \\ -m_{12}\Omega^2 + ic_{12}\Omega + k_{12} & -m_{22}\Omega^2 + ic_{22}\Omega + k_{22} \end{bmatrix} \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

can be rewritten as,

$$\boxed{[Z(i\Omega)]\{\bar{X}\} = \{\bar{F}\}}$$

from which the solution for $\{\bar{X}\}$ is:

$$\boxed{\{\bar{X}\} = \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = [Z(i\Omega)]^{-1}\{\bar{F}\}}$$

where

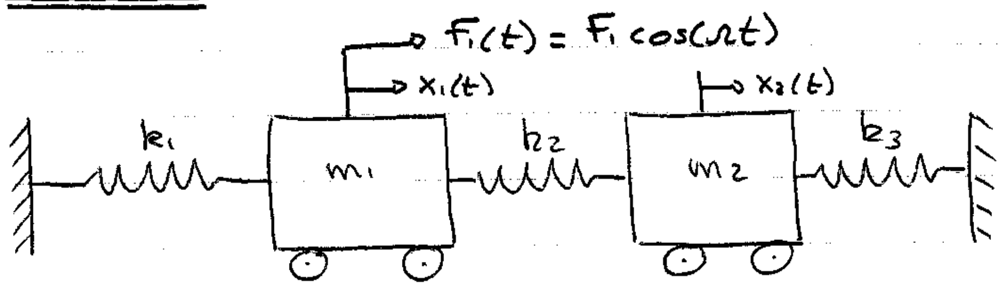
$$[Z(i\Omega)]^{-1} = \frac{1}{Z_{11}(i\Omega)Z_{22}(i\Omega) - Z_{12}^2(i\Omega)} \begin{bmatrix} Z_{22}(i\Omega) & -Z_{12}(i\Omega) \\ -Z_{12}(i\Omega) & Z_{11}(i\Omega) \end{bmatrix}$$

Then,

$$\boxed{\begin{aligned} \bar{X}_1 &= \frac{F_1 Z_{22}(i\Omega) - F_2 Z_{12}(i\Omega)}{Z_{11}(i\Omega)Z_{22}(i\Omega) - Z_{12}^2(i\Omega)} \\ \bar{X}_2 &= \frac{-F_1 Z_{12}(i\Omega) + F_2 Z_{11}(i\Omega)}{Z_{11}(i\Omega)Z_{22}(i\Omega) - Z_{12}^2(i\Omega)} \end{aligned}}$$

The expressions for the two-degree of freedom system are given above. However, the solution procedure is general and it can be applied to any degree of freedom system.

Exercice



Data: $k_1 = k_2 = k_3 = k$

$m_1 = m_2 = m$

Solution Procedure:

① Equations of Motion:

$$\left[\begin{array}{c|c} m & 0 \\ \hline 0 & m \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \left[\begin{array}{c|c} 2k & -k \\ \hline -k & 2k \end{array}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \cos(\Omega t) \\ 0 \end{Bmatrix}$$

② Solve undamped free-vibration problem.

$$\left[\begin{array}{c|c} m & 0 \\ \hline 0 & m \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \left[\begin{array}{c|c} 2k & -k \\ \hline -k & 2k \end{array}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume harmonic solution $\boxed{x_j(t) = \underline{X}_j e^{i\lambda t}}$

$\hookrightarrow$
$$e^{i\lambda t} \left[\begin{array}{c|c} -m\lambda^2 + 2k & -k \\ \hline -k & -m\lambda^2 + 2k \end{array}\right] \begin{Bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(continues next page ...)

The non-trivial solution ($\bar{X}_1 \neq \bar{X}_2 \neq 0$) exists when

$$\det \begin{bmatrix} -m\lambda^2+2k & -k \\ -k & -m\lambda^2+2k \end{bmatrix} = 0$$

$$\Rightarrow \quad (-m\lambda^2+2k)^2 - k^2 = 0$$

$$\Rightarrow \quad 4k^2 + m^2\lambda^4 - 4mk\lambda^2 - k^2 = 0$$

$$\Rightarrow \quad \boxed{m^2\lambda^4 - 4mk\lambda^2 + 3k^2 = 0}$$

Solving the bi-quadratic equation,

$$\lambda^2_{1,2} = \begin{Bmatrix} k/m \\ 3k/m \end{Bmatrix}, \quad \begin{aligned} \lambda_1^2 &= \omega_1^2 = \frac{k}{m} \\ \lambda_2^2 &= \omega_2^2 = \frac{3k}{m} \end{aligned} \quad \underline{\text{Natural frequencies}}$$

③ <u>Solve the damped forced-vibration problem</u>

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \cos(\Omega t) \\ 0 \end{Bmatrix}$$

$Re(Fe^{i\Omega t})$

Using impedance notation

$$\bar{X}_1 = \frac{F_1 Z_{22}(i\Omega) - F_2 Z_{12}(i\Omega)}{Z_{11}(i\Omega) Z_{22}(i\Omega) - Z_{12}^2(i\Omega)}$$

$$\bar{X}_2 = \frac{-F_1 Z_{12}(i\Omega) + F_2 Z_{11}(i\Omega)}{Z_{11}(i\Omega) Z_{22}(i\Omega) - Z_{12}^2(i\Omega)}$$

where

$$Z_{11}(i\Omega) = Z_{22}(i\Omega) = -m\Omega^2 + 2k$$

$$Z_{12}(i\Omega) = -k$$

$$F_2 = 0$$

$$F_1 = F_1$$

Hence,

$$\bar{X}_1 = \frac{(-\Omega^2 m + 2k)F_1}{(-\Omega^2 m + 2k)^2 - k^2} = \frac{(-\Omega^2 m + 2k)F_1}{(-\Omega^2 m + 3k)(-\Omega^2 m + k)}$$

Using $\omega_1^2 = \frac{k}{m}$ and $\omega_2^2 = \frac{3k}{m}$

$$\Rightarrow \bar{X}_1 = \frac{F_1 m(-\Omega^2 + 2k/m)}{m^2(-\Omega^2 + \frac{3k}{m})(-\Omega^2 + \frac{k}{m})} = \frac{F_1}{m}\,\frac{(2\omega_1^2 - \Omega^2)}{(\omega_2^2 - \Omega^2)(\omega_1^2 - \Omega^2)}$$

$$\Rightarrow \bar{X}_1 = \frac{F_1}{m}\,\frac{\cancel{\omega_1^2}\left(2 - \left(\frac{\Omega}{\omega_1}\right)^2\right)}{\underset{\frac{k}{m}}{\cancel{\omega_1^{4}}}\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\Omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\Omega}{\omega_1}\right)^2\right]}$$

$$\Rightarrow \boxed{\bar{X}_1 = \frac{F_1}{k}\,\frac{\left[2 - \left(\frac{\Omega}{\omega_1}\right)^2\right]}{\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\Omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\Omega}{\omega_1}\right)^2\right]}}$$

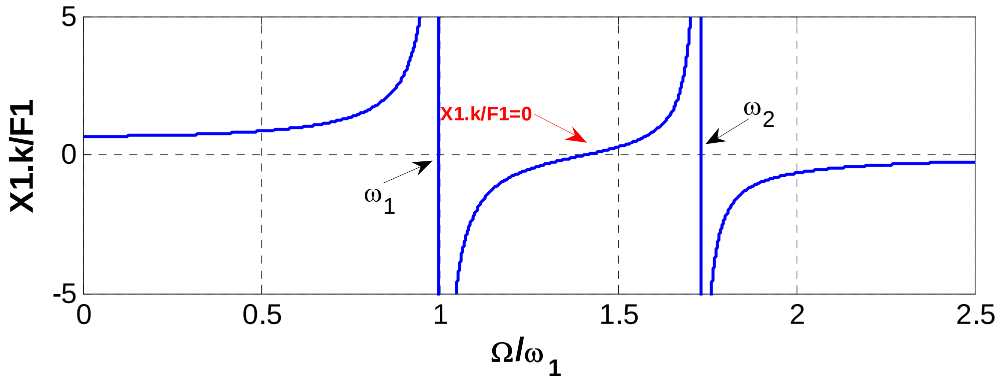
Similarly,

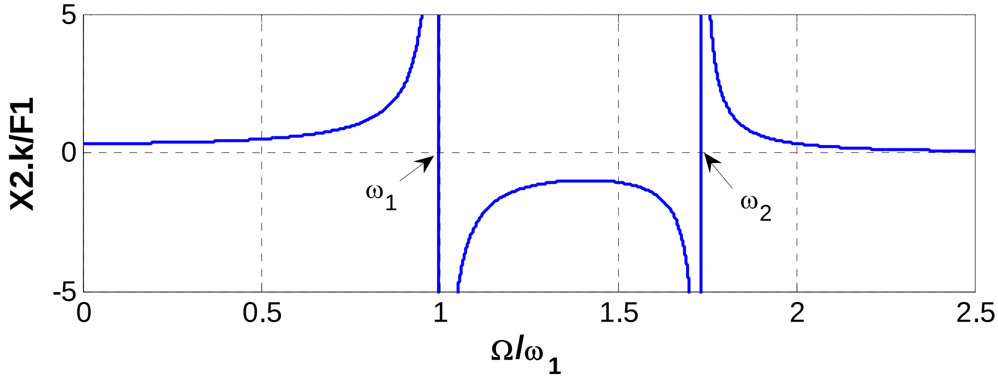
$$\bar{X}_2 = \frac{kF_1}{(-\Omega^2 m + 2k)^2 - k^2} = \frac{kF_1}{(-\Omega^2 m + 3k)(-\Omega^2 m + k)}$$

$$\Rightarrow \bar{X}_2 = \frac{kF_1}{m^2 \underset{\frac{k^2}{m^2}}{\cancel{\omega_1^4}}\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\Omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\Omega}{\omega_1}\right)^2\right]}$$

$$\Rightarrow \boxed{\bar{X}_2 = \frac{F_1}{k}\cdot\frac{1}{\left[\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\Omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\Omega}{\omega_1}\right)^2\right]}}$$

$\hookrightarrow$ See plot of $\frac{X_1 \cdot k}{F_1}$ Vs $\frac{\Omega}{\omega_1}$ and $\frac{X_2 \cdot k}{F_1}$ Vs $\frac{\Omega}{\omega_1}$ on the attached page.





It can be observed from the above figure that the amplitudes X1 and X2 become infinite at $\Omega = \omega_1$ and $\Omega = \omega_2$ which correspond to the two resonances frequencies. At all other values of Ω, the amplitudes of vibrations are finite.

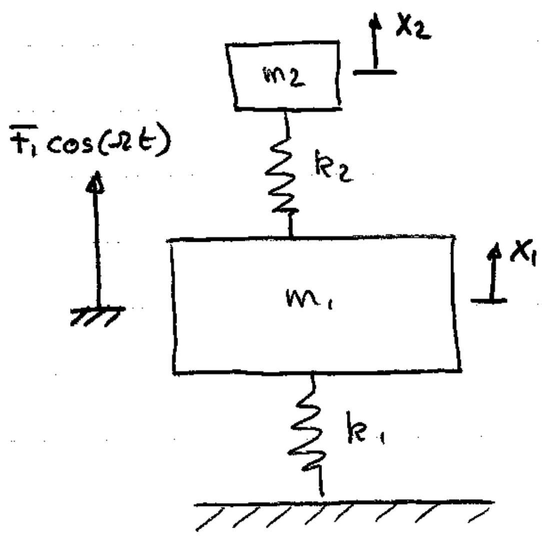
It can also be notes that there is a particular value of the frequency Ω at which the vibration of the first mass m_1=m to which the force F_1 is applied, is reduced to zero. This characteristic forms the basis of the dynamic vibration absorbers.

5. Vibration Isolators

5.1 Undamped Vibration Absorbers

When rotating machinery operates at a constant frequency close to resonance, violent vibrations are induced. Assuming that the system can be represented by a single-degree of freedom, the vibrations may be alleviated by changing either the mass or the spring. At times, however it is not possible. In such case, a second mass and spring can be added to the system, where the added mass-spring are so designed as to produce a two-degree of freedom system whose frequency response is zero at the excitation frequency.

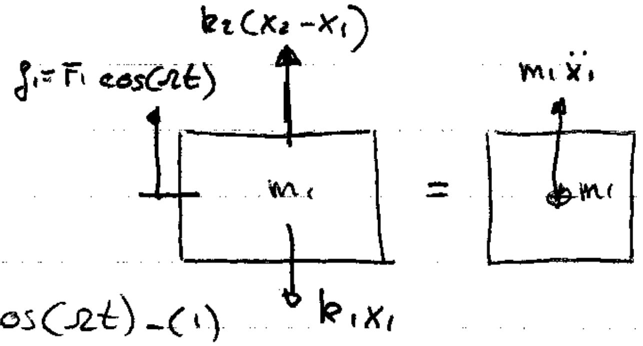
Consider the following system.



ME-400

Equations of motion:

Mass m_1

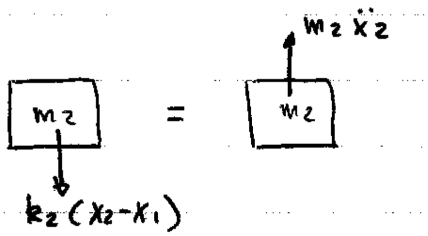


$\Sigma F_x = m_1\ddot{x}_1$ $+ F_1\cos(\Omega t)$

$\Rightarrow \quad -k_1 x_1 + k_2(x_2 - x_1) = m_1\ddot{x}_1$

$\Rightarrow \quad m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = f_1\cos(\Omega t) \quad -(1)$

Mass m_2



$\Sigma F_x = m_2\ddot{x}_2$

$\Rightarrow \quad -k_2(x_2 - x_1) = m_2\ddot{x}_2$

$\Rightarrow \quad m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \quad -(2)$

Or combining (1) and (2) in matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1\cos(\Omega t) \\ 0 \end{Bmatrix}$$

Assuming harmonic solutions:

$$\boxed{\{\bar{x}\} = \{\bar{X}\}e^{i\Omega t} \quad \text{and} \quad \{\bar{f}\} = \{\bar{F}\}e^{i\Omega t}}$$

where in the present case,

$$\begin{cases} x_1 = X_1 e^{i\Omega t} \\ x_2 = X_2 e^{i\Omega t} \\ f_1 = F_1 \, Re(e^{i\Omega t}) \end{cases}$$

Substituting into the equation of motion,

$$\cancel{Re(e^{i\Omega t})}\begin{bmatrix} -\Omega^2 m_1 + (k_1+k_2) & -k_2 \\ -k_2 & -\Omega^2 m_2 + k_2 \end{bmatrix} \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \cancel{Re(e^{i\Omega t})}$$

Using the impedance notation,

$$[Z_{RS}(\Omega)]\{\bar{X}\} = \{\bar{F}\}$$

$$\Rightarrow \quad \{\bar{X}\} = [Z_{RS}]^{-1}\{\bar{F}\}$$

where
$$\begin{cases} Z_{11} = -\Omega^2 m_1 + (k_1+k_2) \\ Z_{12} = -k_2 \\ Z_{22} = -\Omega^2 m_2 + k_2 \end{cases}$$

$$\hookrightarrow [Z_{RS}]^{-1} = \frac{1}{[-\Omega^2 m_1 + (k_1+k_2)][-\Omega^2 m_2 + k_2] - k_2^2} \begin{bmatrix} -\Omega^2 m_2 + k_2 & k_2 \\ k_2 & -\Omega^2 m_1 + (k_1+k_2) \end{bmatrix}$$

$$\hookrightarrow \quad \bar{X}_1 = \frac{F_1(-\Omega^2 m_2 + k_2)}{[-\Omega^2 m_1 + (k_1+k_2)][-\Omega^2 m_2 + k_2] - k_2^2}$$

$$\bar{X}_2 = \frac{k_2 F_1}{[-\Omega^2 m_1 + (k_1+k_2)][-\Omega^2 m_2 + k_2] - k_2^2}$$

It is custumary to introduce the notation:

$$\omega_n = \sqrt{\frac{k_1}{m_1}} \equiv \text{Natural frequency of main system } (m_1)$$

$$\omega_a = \sqrt{\frac{k_2}{m_2}} \equiv \text{Natural frequency of absorber alone } (m_2)$$

$X_{st} \equiv$ Static deflection of main system, $X_{st} = F_1/k_1$

$\mu = \frac{m_2}{m_1} \equiv$ Ratio of absorber mass to main mass

With this notation, the response X_1 and X_2 can be rewritten as:

$$\boxed{\begin{aligned} X_1 &= \frac{\left[1-\left(\frac{\Omega}{\omega_a}\right)^2\right] X_{st}}{\left[1+\mu\left(\frac{\omega_a}{\omega_n}\right)^2-\left(\frac{\Omega}{\omega_n}\right)^2\right]\left[1-\left(\frac{\Omega}{\omega_a}\right)^2\right]-\mu\left(\frac{\omega_a}{\omega_n}\right)^2} \quad \text{— } (*) \\[2em] X_2 &= \frac{X_{st}}{\left[1+\mu\left(\frac{\omega_a}{\omega_n}\right)^2-\left(\frac{\Omega}{\omega_n}\right)^2\right]\left[1-\left(\frac{\Omega}{\omega_a}\right)^2\right]-\mu\left(\frac{\omega_a}{\omega_n}\right)^2} \quad \text{— } (**) \end{aligned}}$$

From eqt. $(*)$ $X_1 = 0$ for $\left(\frac{\Omega}{\omega_a}\right)^2 - 1 = 0$

$$\Rightarrow \boxed{\omega_a = \Omega}$$

For $\omega_a = \Omega$, eqt. $(**)$ gives,

$$X_2 = -\frac{X_{st}}{\mu\left(\frac{\omega_a}{\omega_n}\right)^2} = -\frac{F_1/\cancel{k_1}}{\frac{\cancel{m_2}}{\cancel{m_1}}\left(\frac{k_2/\cancel{m_2}}{\cancel{k_1}/\cancel{m_1}}\right)} = -\frac{F_1}{k_2}$$

The response solution for x_2 becomes,

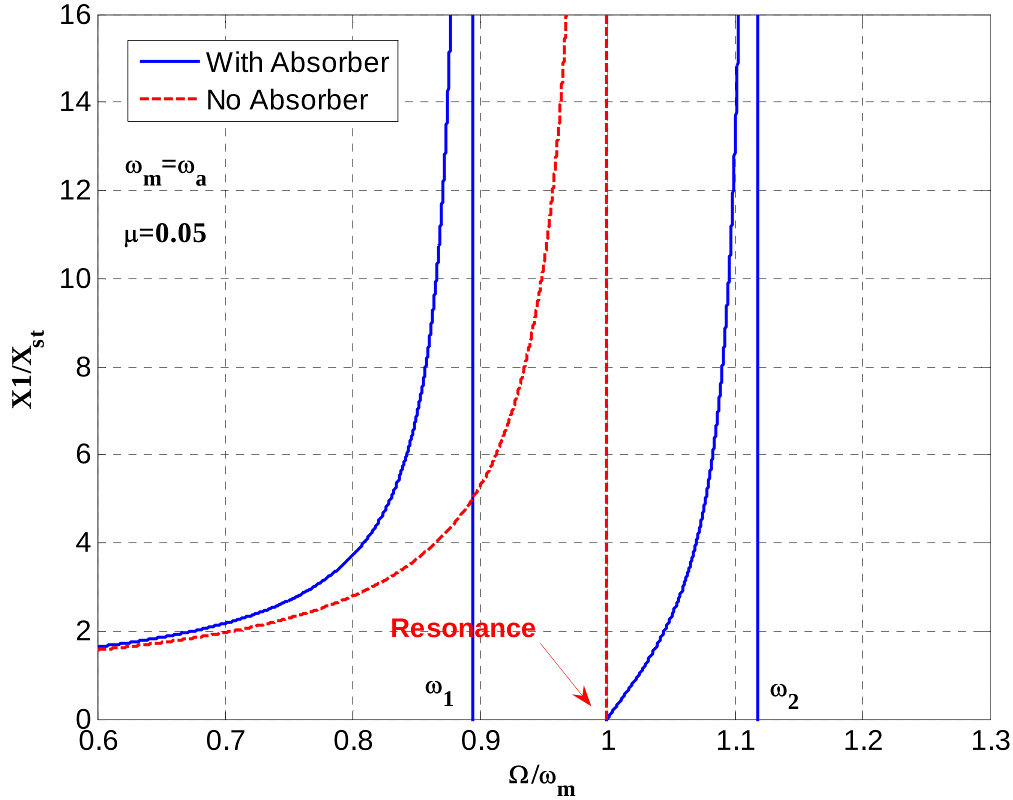
$$x_2 = -\frac{F_1}{k_2} Re(e^{i\Omega t}) = -\frac{F_1}{k_2}\cos(\Omega t)$$

From which the force in the absorber is.

$$\boxed{F_{ab} = k_2 x_2 = -F_1 \cos(\Omega t)}$$

Hence, the absorber exerts on the main mass a force $-F_1 \cos(\Omega t)$ which balances exactly the applied force $F_1 \cos(\Omega t)$.

Although a vibration absorber is designed for a given operating frequency Ω, it can perform satisfactorily for operating frequencies that vary slightly from Ω.



It is observed form the figure above that the addition of the mass-spring (absorber) shifts the two natural frequencies ω_1 and ω_2 from the resonance frequency of the single degree of freedom

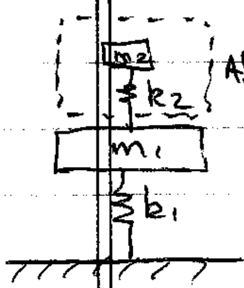
ME-400

Exercice:

A diesel engine weighting 3000 N is supported on a pedestal mount. It has been observed that the engine induces vibration into the surrounding area through its pedestal mount at an operating speed of 6000 RPM.
Determine the parameters (m_2, k_2) of the vibration absorber that will reduce the vibration when mounted on the pedastal. The magnitude of the exciting force is 250 N, and the amplitude of motion of the absorber mass is to be limited to 2 mm.

$$6000 \text{ RPM} \left| \frac{2\pi \text{ Rad}}{1 \text{ Rev}} \right| \cdot \left| \frac{1 \text{ min}}{60 \text{ sec}} \right| = 628.3 \text{ Rad/sec}$$

From the notes:



$$\bar{X}_2 = -\frac{F_1}{k_2}, \quad \omega_a^2 = \frac{k_2}{m_2}$$

magnitude 250 N

$$\Rightarrow \bar{X}_2 = -\frac{F_1}{m_2 \omega_a^2}$$

$$\Rightarrow |m_2| = \left| \frac{F_1}{\bar{X}_2 \omega_a^2} \right|$$

$$\Rightarrow m_2 = \frac{250}{(0.002)(628.3)^2}$$

$$\Rightarrow \boxed{m_2 = 0.31663 \text{ kg}}$$

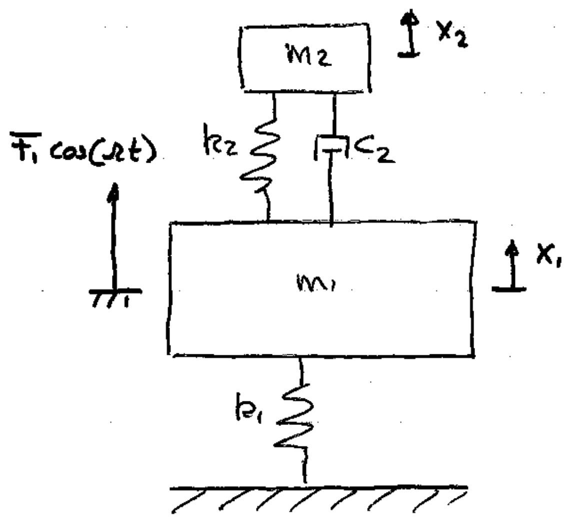
Therefore, $k_2 = \omega_a^2 m_2 \Rightarrow k_2 = 124992.6 \frac{N}{m}$

$$\Rightarrow \boxed{k_2 \simeq 125 \frac{kN}{m}}$$

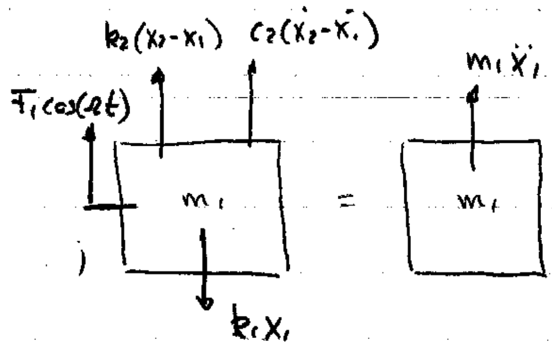
5.2 Damped Vibration Absorbers

The dynamic vibration aborber described in the previous section removes the original resonance peak in the response of the operating machine but introduces two new peaks. As a result, the machine experience large amplitudes (for a very short time) as it passes through the first peak during start-up and stopping. In order to reduce the amplitude of the machine (X_1) a damper can be added.

Consider the following system,



Equations of motion:



Mass m_1

$\Sigma F_x = m_1 \ddot{x}_1$

$\Rightarrow -k_1 x_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) \ldots$

$+ F_1 \cos(\Omega t) = m_1 \ddot{x}_1$

$\Rightarrow m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 + c_2 \dot{x}_1 - c_2 \dot{x}_2 = F_1 \cos(\Omega t) \quad - (i)$

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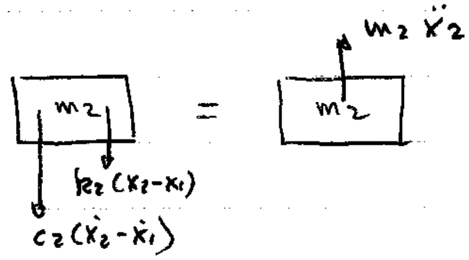
ME-400

Mass m_2



$$\Sigma F_x = m_2\ddot{x}_2$$

$$\Rightarrow -k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) = m_2\ddot{x}_2$$

$$\Rightarrow m_2\ddot{x}_2 - k_2x_1 + k_2x_2 - c_2\dot{x}_1 + c_2\dot{x}_2 = 0 \quad -(ii)$$

Combining (i) and (ii) in matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1\cos(\Omega t) \\ 0 \end{Bmatrix}$$

Assuming harmonic solutions:

$$\boxed{\{\bar{x}\} = \{\bar{X}\}e^{i\Omega t} \text{ and } \{\bar{f}\} = \{\bar{F}\}e^{i\Omega t}}$$

and using the impedance notation

$$\boxed{Z_{rs}(i\Omega) = -m_{rs}\Omega^2 + ic_{rs}\Omega + k_{rs}}$$

Substitution of assumed solutions $\{\bar{x}\}$ and $\{\bar{f}\}$ into the equations of motion yields to

$$\cancel{e^{i\Omega t}}[Z_{rs}(i\Omega)]\{\bar{X}\} = \cancel{e^{i\Omega t}}\{\bar{f}\}$$

$$\Rightarrow \quad [Z_{rs}(i\Omega)]\{\bar{X}\} = \{\bar{f}\}$$

$$\text{OR} \quad \boxed{\{\bar{X}\} = [Z_{rs}(i\Omega)]^{-1}\{\bar{f}\}}$$

ME-400

$$\begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \underbrace{\frac{1}{Z_{22} Z_{11} - Z_{12} Z_{21}} \begin{bmatrix} Z_{22} & -Z_{21} \\ -Z_{12} & Z_{11} \end{bmatrix}}_{[Z_{rs}(i\Omega)]^{-1}} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$\bar{X}_1 = \frac{F_1 Z_{22}}{Z_{22} Z_{11} - Z_{12} Z_{21}} = \frac{F_1 (k_2 - m_2 \Omega^2 + i c_2 \Omega)}{[(k_1 - m_1\Omega^2)(k_2 - m_2\Omega^2) - m_2 k_2 \Omega^2] + i\Omega c_2 (k_1 - m_1\Omega^2 - m_2\Omega^2)}$$

$$\bar{X}_2 = \frac{F_1 Z_{11}}{Z_{22} Z_{11} - Z_{12} Z_{21}} = \cancel{\frac{F_1 (k - m_1\Omega^2 + i c_2 \Omega)}{[(k_1 - m_1\Omega^2)(k_2 - m_2\Omega^2) - m_2 k_2 \Omega^2] + i\Omega c_2 (k_1 - m_1\Omega^2 - m_2\Omega^2)}}$$

X2 =-F1.Z12/Z22Z11-Z12Z21

By defining,

$\omega_n = \sqrt{\frac{k_1}{m_1}}$ ≡ Natural frequency of main system (m_1)

$\omega_a = \sqrt{\frac{k_2}{m_2}}$ ≡ Natural frequency of absorber alone (m_2)

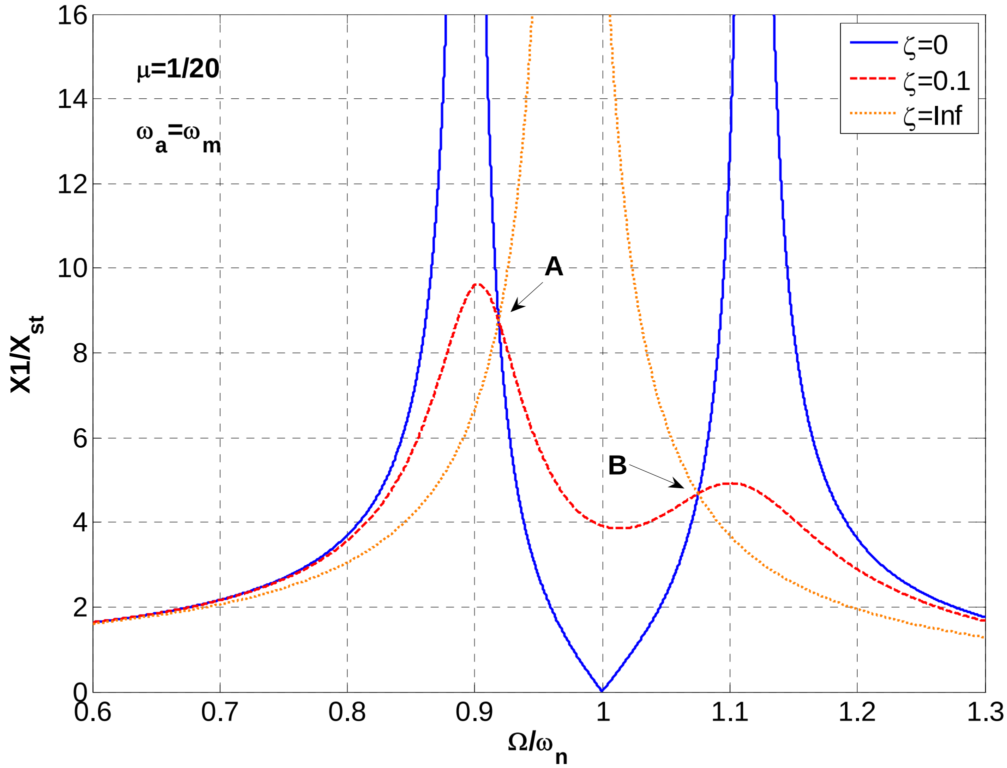
$\mu = \frac{m_2}{m_1}$ ≡ Ratio of absorber mass to main mass.

$\xi = \frac{c_2}{c_c}$ ≡ Damping ratio where $c_c = 2 m_2 \omega_n$

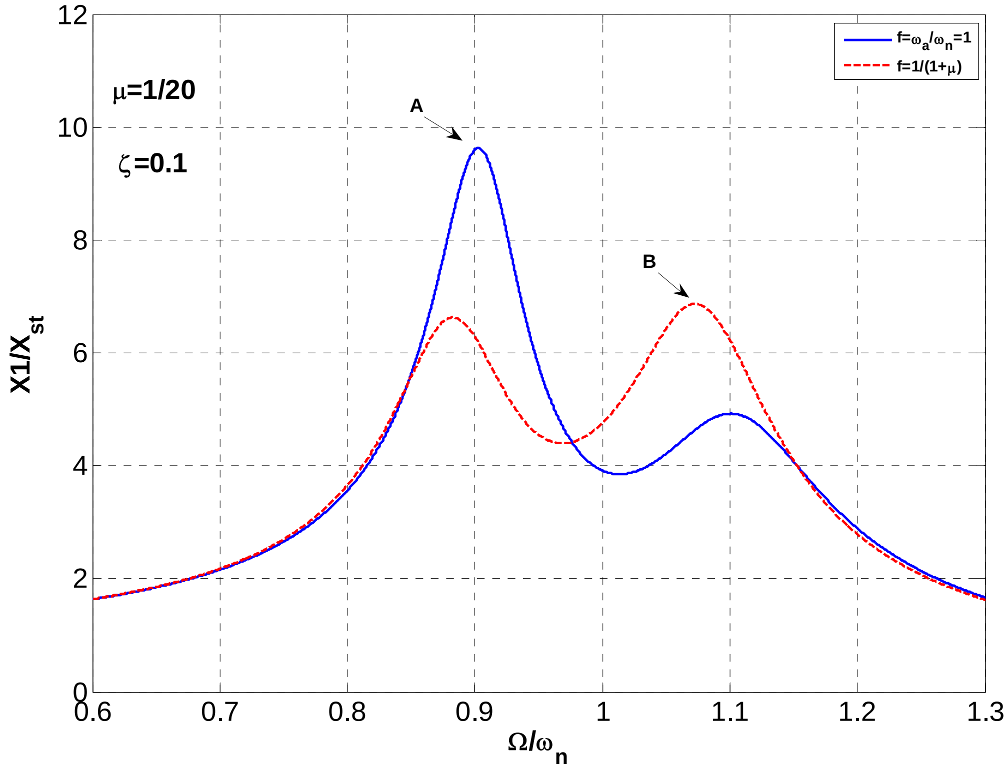
$X_{st} = \frac{F_1}{k_1}$ ≡ static deflection of main system

$f = \frac{\omega_a}{\omega_n}$ ≡ Ratio of natural frequencies

$g = \frac{\Omega}{\omega_n}$ ≡ Forced frequency Ratio



μ=1/20
ω_a=ω_m
ζ=0
ζ=0.1
ζ=Inf
A
B
X1/X_st
Ω/ω_n
16
14
12
10
8
6
4
2
0
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3



μ=1/20
ζ=0.1
A
B
f=ω_a/ω_n=1
f=1/(1+μ)
X1/X_st
Ω/ω_n
0
2
4
6
8
10
12
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3

$$\bar{X}_1 = \left[\frac{X_{st}[(2\xi g)^2 + (g^2 - f^2)^2]}{(2\xi g)^2 (g^2 - 1 + \mu g^2)^2 + [\mu f^2 g^2 - (g^2-1)(g^2-f^2)]^2} \right]^{1/2} \quad -(*)$$

$$\bar{X}_2 = \left[\frac{X_{st}[(2\xi g)^2 + f^4]}{(2\xi g)^2 (g^2 - 1 + \mu g^2)^2 + [\mu f^2 g^2 - (g^2-1)(g^2-f^2)]^2} \right]^{1/2} \quad -(**)$$

Optimally Tuned Vibration Absorber

From the attached figure it can be observed that all curves intersect at points A and B regardless of the damping value. The points A and B can be located by substituting the extreme cases $\xi = 0$ and $\xi = \infty$ into $\bar{X}_1$ (eqt (*)) and equating.

$\Rightarrow$ $$g^4 - 2g^2 \left[\frac{1 + f^2 + \mu f^2}{2+\mu} \right] + \frac{2f^2}{2+\mu} = 0$$

The two roots $g^2_{A,B}$ indicate the values of the frequency ratio $(\cancel{\omega_{A,B}}/\Omega)^2$ corresponding to A and B.

reverse

- It has been observed that the most efficient vibration absorber is one for which $\bar{X}_A \simeq \bar{X}_B$. This condition requires

$$\boxed{f = (1+\mu)^{-1}} \quad \text{OR} \quad \boxed{\frac{\omega_a}{\omega_n} = (1+\mu)^{-1}}$$

- The optimal damping is generally selected as

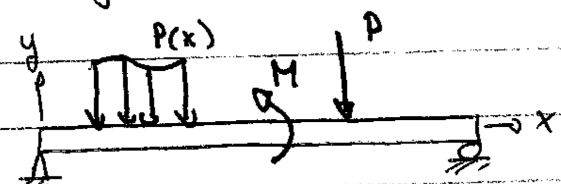
$$\boxed{\xi^2_{optimal} = \frac{3\mu}{8(1+\mu)^3}}$$

ME-400

6. Rayleigh-Ritz Method

The Rayleigh-Ritz method is a technique for the computation of approximate solutions. In structural dynamics it is generally used to determine the eigen-value problem of distributed systems.

Consider the general beam shown below,



Strain-Energy ; $U = \frac{1}{2}\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx$

Kinetic-Energy ; $K = \frac{1}{2}\int_0^L \rho A\left(\frac{dy}{dt}\right)^2 dx \qquad \left(K = \frac{1}{2}mv^2\right)$

$\rho = \frac{m}{V} \Rightarrow m = \rho V$

$V = \rho A dx$.

Assuming Harmonic solution,

$$y(x,t) = Y(x)\, e^{i\lambda t}$$ where $y(x,t)$ is the deflection curve.

Substituting $y(x,t)$ into the strain and kinetic expressions.

$$\begin{cases} U = \frac{1}{2}\int_0^L EI\left(\frac{d^2Y}{dx^2}e^{i\lambda t}\right)^2 dx = e^{2i\lambda t}\cdot\frac{1}{2}\int_0^L EI\left(\frac{d^2Y}{dx^2}\right)^2 dx \\ K = \frac{1}{2}\int_0^L \rho A\left(Y(x)\cdot(i\lambda)e^{i\lambda t}\right)^2 dx. \end{cases}$$

The total energy of the system is,

$$\Pi = U + K = \text{const} = 0$$

$$\Rightarrow \Pi = \left\{ \frac{1}{2}\int_0^L EI\left(\frac{d^2Y}{dx^2}\right)^2 dx - \frac{\lambda^2}{2}\int_0^L \rho A\, Y(x)^2\, dx \right\} e^{2i\lambda t} = 0$$

The above equation is satisfied iff,

① $e^{2i\lambda t} = 0 \Rightarrow$ TRIVIAL SOLUTION

② $$\frac{1}{2}\int_0^L EI\left(\frac{d^2Y}{dx^2}\right)^2 dx - \frac{\lambda^2}{2}\int_0^L \rho A\, Y(x)^2\, dx = 0$$

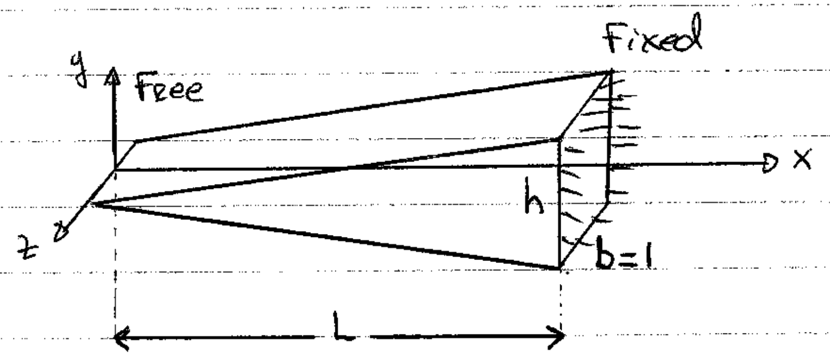
$$\Rightarrow \boxed{\lambda^2 = \omega^2 = \frac{\int_0^L EI\left(\frac{d^2Y(x)}{dx^2}\right)^2 dx}{\int_0^L \rho A\, Y(x)^2\, dx}}$$ Rayleigh Coefficient.

The accuracy of the eigenvalue (natural frequency) depends on the quality of the function $Y(x)$ that must at least satisfy the geometrical boundary conditions.

ME-400

Exercice:

Determine the natural frequency of transverse vibration for the nonuniform cantilever beam shown below:



The cross-section properties are:

$$A(x) = bh\left(\frac{x}{L}\right)$$

$$I(x) = \frac{b}{12}\left[h\left(\frac{x}{L}\right)\right]^3 \quad \text{from} \quad I = \frac{1}{12}bh^3 \text{ for uniform cross-section.}$$

Assuming $Y(x) = \left(1 - \frac{x}{L}\right)^2$

It is important to verify that the function $Y(x)$ satisfies the geometrical boundary conditions.

$$Y(x=L) = 0 \Rightarrow \left(1 - \frac{L}{L}\right)^2 = 0 \rightarrow \underline{\text{SATISFIED}}$$

$$\left.\frac{dY}{dx}\right|_{x=L} = 0 \Rightarrow \frac{d}{dx}\left[1 - \frac{2x}{L} + \left(\frac{x}{L}\right)^2\right]_{x=L} = 0$$

$$\Rightarrow \left. -\frac{2}{L} + \frac{2x}{L^2}\right|_{x=L} = 0 \rightarrow \underline{\text{SATISFIED}}$$

ME-400

The Rayleigh-coefficient equation is

$$\lambda^2 = \omega^2 = \frac{\int_0^L EI(x)\left(\frac{d^2Y(x)}{dx^2}\right)^2 dx}{\int_0^L \rho A(x)\cdot Y(x)^2\, dx}$$

see attached Maple file

$$\omega^2 = \frac{5Eh^2}{2\rho L^4}$$

OR $\boxed{\omega = 1.58114\sqrt{\frac{Eh^2}{\rho L^4}}}$

The exact value for the natural frequency is known to be $1.5343\sqrt{\frac{Eh^2}{\rho L^4}}$. The value given by the Rayleigh-coefficient is 3.05% higher than the exact value.

```
> restart;
> Y:=(1-x/L)^2;
```

$$Y := \left(1 - \frac{x}{L}\right)^2$$

```
> A:=b*h*(x/L);
```

$$A := \frac{b\,h\,x}{L}$$

```
> II:=1/12*b*(h*x/L)^3;
```

$$II := \frac{b\,h^3\,x^3}{12\,L^3}$$

```
> top:=int(E*II*diff(Y,x$2)^2,x=0..L);
>
```

$$top := \frac{E\,b\,h^3}{12\,L^3}$$

```
> bot:=int(rho*A*Y^2,x=0..L);
```

$$bot := \frac{\rho\,b\,h\,L}{30}$$

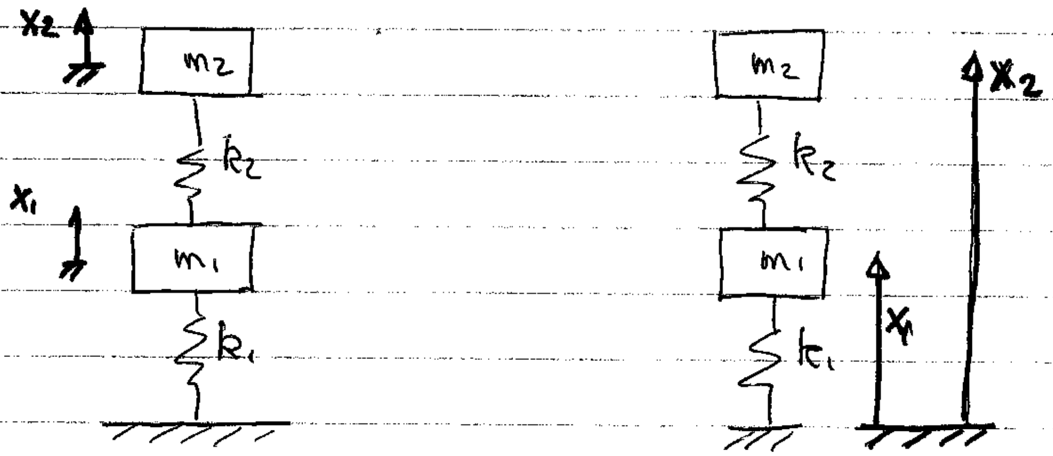
```
> omega:=evalf(sqrt(top/bot));
```

$$\omega := 1.581138830\,\sqrt{\frac{E\,h^2}{L^4\,\rho}}$$

7. Absolute and Relative Coordinates

In all equations of motion derived in the previous sections, the coupling between a two degree of freedom system is made on a stiffness matrix [K]. However, the equations of motions can be derived in such a way that the coupling is made on the mass matrix [M].

Consider the following two degree of freedom system



RELATIVE COORDINATES

ABSOLUTE COORDINATES

For each coordinate system, let us derive the equations of motion using the Lagrange's method

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$ Lagrange's Equation

where $L = KE - PE$

ME - 400

RELATIVE COORDINATES

$$KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$L = KE - PE$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2$$

Substituting into the Lagrange's expression term by term,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial L}{\partial x_1} = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial L}{\partial x_2} = -k_2 (x_2 - x_1)$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \quad \text{where } i = 1, 2}$$

FOR $i = 1$ $$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

FOR $i = 2$ $$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

ME-400

or in matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\text{Elastic coupling}} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- **ABSOLUTE COORDINATES**

$$KE = \frac{1}{2} m_1 \upsilon_1^2 + \frac{1}{2} m_2 (\upsilon_1^2 + \upsilon_2)^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2$$

$$PE = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$$

Substituting into the Lagrange's expression,

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \quad \text{where } i = 1,2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 + m_2 (\ddot{x}_1 + \ddot{x}_2)$$

$$\frac{\partial L}{\partial x_1} = -k_1 x_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 (\ddot{x}_1 + \ddot{x}_2)$$

$$\frac{\partial L}{\partial x_2} = -k_2 x_2$$

For $i=1$: $(m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_2 + k_1 x_1 = 0$

For $i=2$: $m_2 \ddot{x}_1 + m_2 \ddot{x}_2 + k_2 x_2 = 0$

or in matrix form,

$$\boxed{\underbrace{\left[\begin{array}{c|c} m_1+m_2 & m_2 \\ \hline m_2 & m_2 \end{array}\right]}_{\text{Inertial coupling}} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \left[\begin{array}{c|c} k_1 & 0 \\ \hline 0 & k_2 \end{array}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Both system of equations of motion represent the same problem but have been derived in different coordinate systems. However, both must yield to the same eigenvalue-eigenvector results.

In order to check the above statement, let us solve both eigenvalue/eigenvector problem for the data below:

$$m_1 = 1\,kg \qquad k_1 = 10\,N/m$$
$$m_2 = 2\,kg \qquad k_2 = 20\,N/m$$

- Relative coordinates

$$\left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \left[\begin{array}{c|c} k_1+k_2 & -k_2 \\ \hline -k_2 & k_2 \end{array}\right] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 2 \end{array}\right] \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 30 & -20 \\ -20 & 20 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming harmonic solution $\{x_j\} = X_1 e^{i\lambda t} \quad j = 1,2$

ME-400

$$e^{i\lambda t}\left[\begin{array}{c|c} -\lambda^2+30 & -20 \\ \hline -20 & -2\lambda^2+20 \end{array}\right]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The above equation is satisfied iff,

- $e^{i\lambda t} = 0 \Rightarrow$ TRIVIAL SOLUTION
- OR $\det\left(\left[\begin{array}{c|c} -\lambda^2+30 & -20 \\ -20 & -\lambda^2+20 \end{array}\right]\right) = 0$

$$\Rightarrow \boxed{(-\lambda^2+30)(-\lambda^2+20) - 20^2 = 0}$$

Solving the characteristic equation for $\lambda_{1,2}^2 = \omega_{1,2}^2$

$$\omega_1 = \sqrt{\lambda_1^2} = 1.639 \quad \text{Rad/sec}$$
$$\omega_2 = \sqrt{\lambda_2^2} = 6.109 \quad \text{Rad/sec}$$

The eigenvectors are:

MODE 1 ① $\underline{\lambda = \lambda_1^2}$

$$\left[\begin{array}{c|c} -\lambda_1^2+30 & -20 \\ -20 & -2\lambda_1^2+20 \end{array}\right]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} (-\lambda_1^2+30) X_1 - 20 X_2 = 0 \\ -20 X_1 + (-2\lambda_1^2+20) X_2 = 0 \end{cases}$$

$$\Rightarrow \frac{X_1^{(1)}}{X_2^{(1)}} = \frac{20}{(-\lambda_1^2+30)} = \frac{(-2\lambda_1^2+20)}{20} = 0.732$$

ME-400

If $\bar{X}_1^{(1)} = 1 \Rightarrow \bar{X}^{(1)} = \begin{pmatrix} 1 \\ 1.366 \end{pmatrix}$ eigenvector for $\lambda_1^2 = 2679$

MODE 2

② $\underline{\lambda = \lambda_2^2}$

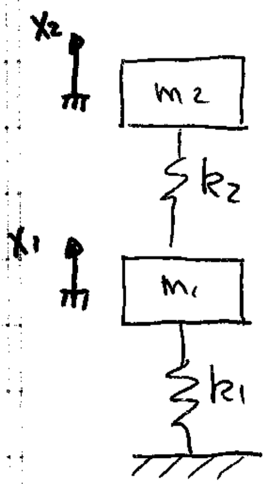
$$\left[\begin{array}{c|c} -\lambda_2^2 + 30 & -20 \\ \hline -20 & -2\lambda_2^2 + 20 \end{array}\right] \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

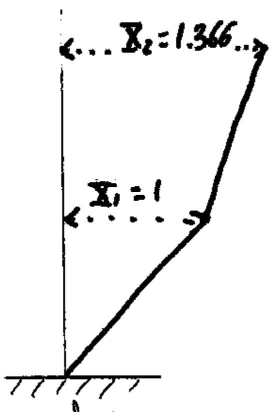
$$\hookrightarrow \quad \frac{\bar{X}_1^{(2)}}{\bar{X}_2^{(2)}} = \frac{20}{(-\lambda_2^2 + 30)} = \frac{(-2\lambda_2^2 + 20)}{20} = -2.732$$

If $\bar{X}_1^{(2)} = 1 \Rightarrow \bar{X}^{(2)} = \begin{pmatrix} 1 \\ -0.366 \end{pmatrix}$ eigenvector for $\lambda_2^2 = 37.3205$

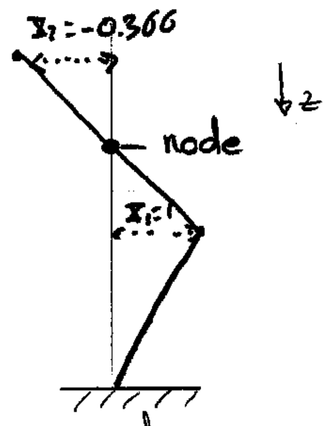
The mode-shape (eigenvector) is:

$$\bar{X} = [\Phi] = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1.366 & -0.366 \end{array}\right] = [\text{mode 1} \;\; \text{mode 2}]$$





mode 1



mode 2

Position of node:

$$\frac{0.366}{z} = \frac{1}{(1-z)} \Rightarrow \boxed{z = 0.268}$$

ME-400

- Absolute coordinates

$$\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

OR

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming harmonic solution

$$\hookrightarrow \quad e^{i\lambda t} \begin{bmatrix} -3\lambda^2+10 & -2\lambda^2 \\ -2\lambda^2 & -2\lambda^2+20 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Other than the trivial solution ($e^{i\lambda t}=0$), solutions to the above equation are obtained from the roots ($\lambda_{1,2}^2$) of the characteristic equation.

$$(-3\lambda^2+10)(-2\lambda^2+20) - 4\lambda^4 = 0$$

$$\omega_1 = \sqrt{\lambda_1^2} = 1.639 \text{ rad/sec}$$
$$\omega_2 = \sqrt{\lambda_2^2} = 6.109 \text{ rad/sec}$$

The eigenvectors are.

MODE 1 ① $\underline{\lambda = \lambda_1^2}$

$$\begin{bmatrix} -3\lambda_1^2+10 & -2\lambda_1^2 \\ -2\lambda_1^2 & -2\lambda_1^2+20 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\hookrightarrow \quad \frac{X_1^{(1)}}{X_2^{(1)}} = \frac{-2\lambda_1^2}{-3\lambda_1^2+10} = \frac{-2\lambda_1^2+20}{-2\lambda_1^2} = -2.732$$

FOR $\bar{X}_1^{(1)} = 1 \Rightarrow \bar{X}^{(1)} = \begin{pmatrix} 1 \\ 0.366 \end{pmatrix}$

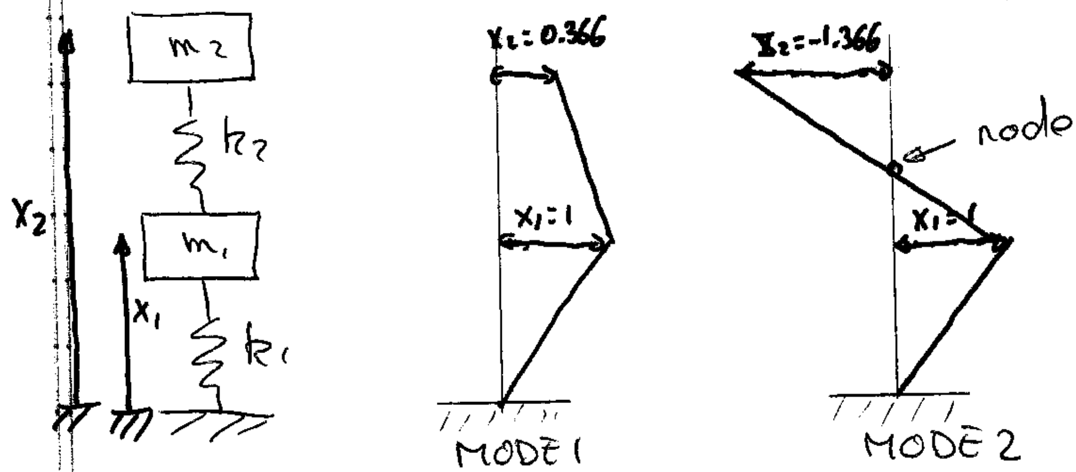
MODE 2

② $\lambda = \lambda_2^2$

$$\begin{bmatrix} -3\lambda_2^2 + 10 & -2\lambda_2^2 \\ -2\lambda_2^2 & -2\lambda_2^2 + 20 \end{bmatrix} \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \frac{\bar{X}_1^{(1)}}{\bar{X}_2^{(1)}} = \frac{-2\lambda_2^2}{-3\lambda_2^2 + 10} = \frac{-2\lambda_2^2 + 20}{-2\lambda_2^2} = 0.732$$

FOR $\bar{X}_1^{(2)} = 1 \Rightarrow \bar{X}^{(2)} = \begin{pmatrix} 1 \\ -1.366 \end{pmatrix}$



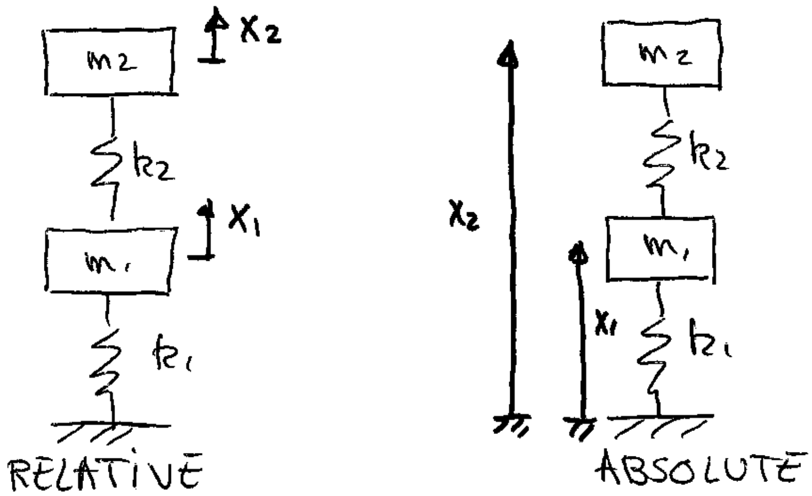
$$\bar{X} = [\phi] = \begin{bmatrix} 1 & 1 \\ 0.366 & -1.366 \end{bmatrix}$$

Note: Relative and absolute give identical eigenvalues (frequencies) but not identical eigenvector $\Rightarrow$ why?

(Answer on next page)

ME-400

The eigenvectors are also identical but are measured from different coordinate system.



$$[\phi]_{REL} = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1.366 & -0.366 \end{array}\right] \qquad [\phi]_{ABS} = \left[\begin{array}{c|c} 1 & 1 \\ \hline 0.366 & -1.366 \end{array}\right]$$

One can easily observe that if we keep $\Phi_1 = 1$, the relation to go from RELATIVE to ABSOLUTE coordinates is $(\Phi_2)_{ABS} = (\Phi_2)_{REL} + 1$

```
% ME-400
% Absolute & Realtives Coordinates
clc
clear all


% Data:
m1=1;
m2=2;
k1=10;
k2=20;


% Relative Coordinates

disp('RELATIVE COORDINATES')
M=[m1 0;0 m2]
K=[k1+k2 -k2;-k2 k2]

[vect val]=eig(K,M);

fprintf('\n Eigen-value omega_1=%g rad/sec\n',sqrt(val(1,1)))
fprintf('\n Eigen-value omega_2=%g rad/sec\n',sqrt(val(2,2)))


Phi_1=vect(:,1)/vect(1,1);
Phi_2=vect(:,2)/vect(1,2);
fprintf('\n Eigenvector=Mode-Shape:\n')
Phi=[Phi_1 Phi_2]
disp('====================================================================')
% Absolute Coordinates
disp('ABSOLUTE COORDINATES')
M=[m1+m2 m2;m2 m2]
K=[k1 0;0 k2]

[vect val]=eig(K,M);

fprintf('\n Eigen-value omega_1=%g rad/sec\n',sqrt(val(1,1)))
fprintf('\n Eigen-value omega_2=%g rad/sec\n',sqrt(val(2,2)))


Phi_1=vect(:,1)/vect(1,1);
Phi_2=vect(:,2)/vect(1,2);
fprintf('\n Eigenvector=Mode-Shape:\n')
Phi=[Phi_1 Phi_2]
```

```
RELATIVE COORDINATES

M =

     1     0
     0     2


K =

    30   -20
   -20    20


 Eigen-value omega_1=1.63692 rad/sec

 Eigen-value omega_2=6.10905 rad/sec

 Eigenvector=Mode-Shape:

Phi =

    1.0000    1.0000
    1.3660   -0.3660

=================================================================
ABSOLUTE COORDINATES

M =

     3     2
     2     2


K =

    10     0
     0    20


 Eigen-value omega_1=1.63692 rad/sec

 Eigen-value omega_2=6.10905 rad/sec

 Eigenvector=Mode-Shape:

Phi =

    1.0000    1.0000
    0.3660   -1.3660

>>
```